GRADIENT THEORY FOR PLASTICITY VIA HOMOGENIZATION OF DISCRETE DISLOCATIONS

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ABSTRACT. In this paper, we deduce a macroscopic strain gradient theory for plasticity from a model of discrete dislocations.

We restrict our analysis to the case of a cylindrical symmetry for the crystal in exam, so that the mathematical formulation will involve a two dimensional variational problem.

The dislocations are introduced as point topological defects of the strain fields, for which we compute the elastic energy stored outside the so called core region. We show that the Γ-limit as the core radius tends to zero and the number of dislocations tends to infinity of this energy (suitably rescaled), takes the form

$$E = \int_{\Omega} \left(W(\beta^e) + \varphi(\operatorname{Curl} \beta^e) \right) dx,$$

where β^e represents the elastic part of the macroscopic strain, and $\operatorname{Curl} \beta^e$ represents the geometrically necessary dislocation density. The plastic energy density φ is defined explicitly through an asymptotic cell formula, depending only on the elastic tensor and the class of the admissible Burgers vectors, accounting for the crystalline structure. It turns out to be positively 1-homogeneous, so that concentration on lines is permitted, accounting for the presence of pattern formations observed in crystals such as dislocation walls.

Keywords : variational models, energy minimization, relaxation, plasticity, strain gradient theories, stress concentration, dislocations.

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CONTENTS

1. INTRODUCTION

In the last decade the study of crystal defects such as *dislocations* has been increasingly active. The presence of dislocations (and their motion) is indeed considered the main mechanism of plastic

deformations in metals. Various phenomenological models have been proposed to account for plastic effects due to dislocations, such as the so called *strain gradient* theories.

The aim of this paper is to provide a rigorous derivation of a strain gradient theory for plasticity as a mesoscopic limit of systems of discrete dislocations, which are introduced as point defects of the strain fields, for which we compute the elastic energy stored outside the so called *core region*.

We focus on the stored energy, disregarding dissipation, and we restrict our analysis to the case of a cylindrical symmetry for the crystal in exam, so that the mathematical formulation will involve a two dimensional variational problem.

Many theories of plasticity are framed within linearized elasticity. In classical linear elasticity a displacement of Ω is a regular vector field $u : \Omega \to \mathbb{R}^2$. The equilibrium equations have the form $\mathrm{Div}\,\mathbb{C}[e(u)] = 0$, with $\mathbb{C}$ a linear operator from $\mathbb{R}^{2\times2}$ into itself, and $e(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top)$ the infinitesimal strain tensor. The corresponding *elastic energy* is

$$(1) \qquad\qquad \int_\Omega W(\nabla u)\,dx,$$

where

$$(2) \qquad\qquad W(\xi) = \frac{1}{2}\mathbb{C}\xi : \xi,$$

is the *elastic energy density*, and the elasticity tensor $\mathbb{C}$ satisfies

$$(3) \qquad\qquad c_1|\xi^{\mathrm{sym}}|^2 \leq \mathbb{C}\xi : \xi \leq c_2|\xi^{\mathrm{sym}}|^2 \qquad \text{for any } \xi \in \mathbb{M}^{2\times2}\,,$$

where $\xi^{\mathrm{sym}} := \frac{1}{2}(\xi + \xi^\top)$ and c_1 and c_2 are two given positive constants. In this linear framework the presence of plastic deformations is classically modeled by the additive decomposition of the gradient of the displacement in the elastic strain β^e and the plastic strain β^p, i.e., $\nabla u = \beta^e + \beta^p$. The elastic energy induced by a given plastic strain is then

$$(4) \qquad\qquad \int_\Omega W(\nabla u - \beta^p)\,dx\,.$$

In view of their microscopic nature the presence of dislocations, responsible for the plastic deformation, can be modeled in the continuum by assigning the Curl of the field β^p. The quantity $\mathrm{Curl}\,\beta^p = \mu$ is then called the *Nye's dislocation density tensor*, (see [19]). Inspired by this idea, the strain gradient theory for plasticity, first proposed by Fleck and Hucthinson [10] and then developed by Gurtin (see for instance [13], [14]), assigns an additional phenomenological energy to the plastic deformation depending on the dislocation density, so that the stored energy looks like

$$(5) \qquad\qquad \int_\Omega W(\nabla u - \beta^p)\,dx + \int_\Omega \varphi(\mathrm{Curl}\,\beta^p)\,dx\,.$$

Note that for a given preexisting strain β^p, the minimum of the energy in (4) (possibly subject to external loads) depends only on $\mathrm{Curl}\,\beta^p$. Hence the relevant variable in this problem is a strain field β whose Curl is prescribed. The main issue with this model is the choice of the function φ in (5). In fact the usual choice (see for instance [15]) is to take φ quadratic, and then to fit the free parameters through experiments as well as simulations, as in [18]. This choice has the well-known disadvantage of preventing concentration of the dislocation density, that is instead allowed, for instance, by the choice $\varphi(\mathrm{Curl}\,\beta) := |\mathrm{Curl}\,\beta|$, recently proposed by Conti and Ortiz in [7] (see also [20]).

The aim of this paper is to derive the strain gradient model (5) for the stored energy, starting from a basic model of discrete dislocations that accounts for the crystalline structure. As a consequence we will also get a formula for the function φ, determined only by the elasticity tensor $\mathbb{C}$ and the Burgers vectors of the crystal.

A purely discrete description of a crystal in the presence of dislocations can be given after introducing the discrete equivalent of β^p and $\mu := \mathrm{Curl}\,\beta^p$, following the approach of Ariza and Ortiz [1]. That model is the starting point for a microscopic description of dislocations. In the passage from discrete to continuum one can consider a intermediate (so to say) semi-discrete model known in literature as *discrete dislocation model* (see also [9], [6]), in which the atomic scale is introduced as an internal small parameter ε referred to as the *core radius*. The gap between the purely discrete model and the discrete dislocation model can be usually filled through an interpolation procedure (see for instance

[21]). In this paper, to simplify matter, we will adopt the discrete dislocation model.

In the discrete dislocation model a straight dislocation orthogonal to the cross section Ω is identified with a point $x_0 \in \Omega$ or, more precisely, with a small region surrounding the dislocation referred to as *core region*, i.e., a ball $B_\varepsilon(x_0)$, being the core radius ε proportional to (the ratio between the dimensions of the crystal and) the underlying lattice spacing. The presence of the dislocation can be detected looking at the topological singularities of a continuum strain field β, i.e.,

$$\int_{\partial B_\varepsilon(x_0)} \beta \cdot t \, ds = b \, ,$$

where t is the tangent to $\partial B_\varepsilon(x_0)$ and b is the Burgers vector of the given dislocation.

We will focus our analysis to the case when there may be many dislocations crossing the domain Ω. A generic distribution of N dislocations will be then identified with N points $\{x_i\}_{1,...,N}$ (or equivalently with the corresponding core regions), each corresponding to some Burgers vectors b_i belonging to a finite set of *admissible Burgers vectors* S, depending on the crystalline structure, e.g., for square crystals, up to a renormalization factor, $S = \{e_1, e_2, -e_1, -e_2\}$.

In our analysis we introduce a second small scale $\rho_\varepsilon \gg \varepsilon$ (the *hard core radius*) at which a cluster of dislocations will be identified with a multiple dislocation. In mathematical terms this corresponds to introducing the span $\mathbb{S}$ of S on $\mathbb{Z}$ (i.e., the set of finite combinations of Burgers vectors with integer coefficients) and to representing a generic distribution of dislocations as a measure μ of the type

$$\mu = \sum_{i=1}^N \delta_{x_i} \xi_i, \qquad \xi_i \in \mathbb{S} \, ,$$

where the distance between the x_i's is at least $2\rho_\varepsilon$. The admissible strain fields β corresponding to μ are defined outside the core region, namely in $\Omega \setminus \cup_i B_\varepsilon(x_i)$, and satisfy

$$(6) \qquad \int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \xi_i \, , \qquad \text{for all} \quad i = 1, ..., N \, .$$

It is well-known (and it will be discussed in detail in the sequel) that the plastic distortion due to the presence of dislocations decays as the inverse of the distance from the dislocations. This justifies the use of the linearized elastic energy outside the core region $\cup_i B_\varepsilon(x_i)$. On the other hand, considerations at a discrete level show that the elastic energy stored in the core region can be neglected. Therefore, the elastic energy corresponding to μ and β is given by

$$(7) \qquad E_\varepsilon(\mu, \beta) := \int_{\Omega_\varepsilon(\mu)} W(\beta) \, dx \, ,$$

where $\Omega_\varepsilon(\mu) = \Omega \setminus \cup_i B_\varepsilon(x_i)$. By minimizing the elastic energy (7) among all strains satisfying (6), we obtain the energy induced by the dislocation μ. Note that, in view of the compatibility condition (6), this residual energy is positive whenever $\mu \neq 0$.

In the case $W(\beta) = |\beta|^2$, with β a curl free vector field in $\Omega \setminus \cup_i B_\varepsilon(x_i)$, this model coincides (setting $\beta = \nabla\theta$, θ being the phase function) with the energy proposed by Bethuel, Brezis and Helein (see [4]) to deduce the so-called renormalized energy in the study of vortices in superconductors. In the context of dislocations this choice of W and β corresponds to a model for screw dislocations in an anti-planar setting. A Γ-convergence result in this framework has been studied in [21] in the energy regime $E_\varepsilon \approx |\log \varepsilon|$ that corresponds to a finite number of dislocations.

In the planar context of edge dislocations, that we consider here, the connection with the Ginzburg-Landau model for vortices is more formal and the analysis has specific difficulties due to the vectorial nature of the circulation constraint (6) and the properties of the energy density W. A first analysis of this model has been perfomed by Cermelli and Leoni in [6] where they obtain an asymptotic expansion of the energy functionals E_ε in (7), in the case of a fixed configuration of dislocations. In particular they deduce the corresponding renormalized energy representing the effect of the Peach-Koehler force.

The purpose of this paper is to consider the energy functional E_ε in (7), where the unknowns are the distribution of dislocations and the corresponding admissible strains, and to study its asymptotic behavior in terms of Γ-convergence as the lattice spacing ε tends to zero and the number of dislocations

N_ε goes to infinity. Under a suitable condition on the hard core scale, we can show that in the asymptotics the energy E_ε can be decomposed into two effects: the *self-energy*, concentrated in the hard core region, and the *interaction energy*, diffused in the remaining part of Ω. The general idea is that the Γ-limit of the (rescaled) energy functionals E_ε as ε goes to zero is given by the sum of these two effects. For different regimes for the number of dislocations N_ε the corresponding Γ-limit will exhibit the dominance of one of the two effects: the self-energy is predominant for $N_\varepsilon \ll |\log \varepsilon|$, while the interaction energy is predominant for $N_\varepsilon \gg |\log \varepsilon|$. In a critical regime $N_\varepsilon \approx |\log \varepsilon|$ the two effects will be balanced and the structure of the limiting energy is the following

$$(8) \qquad \int_\Omega W(\beta)\, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

The first term is the elastic energy of the limiting rescaled strain and comes from the interaction energy. The second term represents the plastic energy and comes from the self-energy; it depends only on the rescaled dislocation density $\mu = \operatorname{Curl}\beta$ through a positively homogeneous of degree 1 density function φ, defined by a suitable asymptotic cell problem formula. The constraint $\operatorname{Curl}\beta = \mu$ comes from the admissibility condition (6) that the admissible strains have to satisfy at a discrete level.

If (β, μ) is a configuration that makes the continuous energy in (8) finite, we have necessarily that β^{sym} belongs to $L^2(\Omega; \mathbb{M}^{2\times 2})$, and μ has finite mass. A natural question arises: does μ belong to the space $H^{-1}(\Omega; \mathbb{R}^2)$? We give a positive answer to this problem proving a Korn type inequality for fields whose curl is a prescribed measure, based on a fine estimate for elliptic systems with L^1-data recently proved by Bourgain, Brezis and Van Schaftingen (see [3], [5]). In particular we deduce that concentration of dislocations on lines is permitted, accounting for the presence of pattern formations observed in crystals such as *dislocation walls*, while concentration on points is not permitted. An additional feature of the limit energy is the anisotropy of the self-energy density inherited from the anisotropic elastic tensor and the class of the admissible Burgers vectors accounting for the crystalline structure.

The idea of deducing continuous models by homogenization of lower dimensional singularities has been used in other contexts. Our result is very much in the spirit of earlier results on the asymptotic analysis for the Ginzburg-Landau model for superconductivity as the number of vortices goes to infinity, done by Jerrard and Soner in [16], and by Sandier and Serfaty in [23] (see also [24] and the references therein).

As for the study of dislocations, a similar analysis was done by Garroni and Focardi ([11]) starting from a phase field model introduced by Koslowski, Cuitino and Ortiz ([17]) and inspired by the Peierls-Nabarro model (see also [12]).

The plan of the paper is the following. In Section 2 we compute heuristically the asymptotic behavior of the self and the interaction energy in terms of the number of dislocations N_ε and the core radius ε. In Section 3 we introduce rigorously the mathematical setting of the problem, defining the class of admissible configurations of dislocations, the class of the corresponding admissible strains and the rescaled energy functionals. In Section 4 we provide the asymptotic cell problem formulas defining the density of the plastic energy φ. In Section 5 we prove a Korn type inequality for fields with prescribed curl. In Section 6 we give our main result concerning the Γ-convergence of the energy functionals in the critical regime $N_\varepsilon \approx |\log \varepsilon|$, while Section 7 and Section 8 are devoted to the sub critical and to the super critical case respectively.

2. Heuristic for the scaling

In this section we identify the energy regimes, as the internal scale ε tends to zero, of the elastic energy induced by a distribution of dislocations $\mu_\varepsilon := \sum_{i=1}^{N_\varepsilon} \xi_i \delta_{x_i}$, where we omit the dependence on ε of the x_i's. The idea is that if the scaling of the energy of a given sequence of distributions of dislocations is assigned, this provides a bound for N_ε. In this section, in order to identify the relevant energy regimes, we do the converse: we assume to know the number of dislocations N_ε present in the crystal, and we compute (heuristically) the corresponding energy behavior as ε goes to zero.

As already mentioned in the introduction, the general idea is that the energy will be always given by the sum of two terms: the *self-energy*, concentrated in a small region (the *hard core region*) surrounding the dislocations, and the *interaction energy*, diffused in the remaining part of the domain Ω. We will

first introduce and compute heuristically the asymptotic behavior of these two terms separately. Then we will identify the leading term (which depends on the behavior of N_ε) between these two energies.

2.1. **Self-energy.** The self-energy of a distribution of dislocations $\mu_\varepsilon := \sum_{i=1}^{N_\varepsilon} \xi_i \delta_{x_i}$ is essentially given by the sum of the energies that would be induced by a single dislocation $\mu_\varepsilon^i := \xi_i \delta_{x_i}$, $i = 1, \dots N_\varepsilon$ (i.e., if no other dislocations $\xi_j \delta_{x_j}$, $j \neq i$ were present in the crystal). As we will see, the self-energy of a single dislocation $b \delta_x$ is asymptotically (as $\varepsilon \to 0$) independent of the position of the dislocation point x in Ω, depending only on the elasticity tensor $\mathbb{C}$ and on the Burgers vector b.

We begin by computing the self-energy in a very simple situation. We assume for the time being that Ω is the ball B_1 of radius one and center zero, that $\mu_\varepsilon \equiv \mu = b\,\delta_0$, with $|b| = 1$, and we compute the energy of an admissible strain β considering the toy energy given by the square of the L^2 norm of β. More precisely, we consider

$$(9) \qquad E_\varepsilon^{\text{toy}}(\mu, \beta) := \int_{B_1 \setminus B_\varepsilon} |\beta|^2 \, dx,$$

and the self-energy induced by μ given by minimizing this energy among all admissible strains β compatible with μ_ε, i.e., satisfying

$$\int_{\partial B_\varepsilon} \beta \cdot t \, ds = b.$$

In this setting the minimum strain $\beta(\mu)$ can be computed explicitly, and in polar coordinates it is given by the expression

$$\beta(\mu)(\theta, r) := \frac{1}{2\pi r} t(r, \theta) \otimes b,$$

where $t(r, \theta)$ is the tangent unit vector to $\partial B_r(0)$ at the point (r, θ). We deduce the following expression for the self-energy

$$(10) \qquad E_\varepsilon^{\text{self}} := \int_\varepsilon^1 \frac{1}{2\pi r} \, dr = \frac{1}{2\pi}(\log 1 - \log \varepsilon).$$

Hence, as $\varepsilon \to 0$ the self-energy of a single dislocation behaves like $|\log \varepsilon|$.

Notice that most of the self-energy is concentrated around a small region surrounding the dislocation. Indeed, fix $s > 0$, and compute the energy stored in the ball $B_{\varepsilon^s}(0)$,

$$\int_{B_{\varepsilon^s}(0)} |\beta(\mu)|^2 \, dx = \int_\varepsilon^{\varepsilon^s} \frac{1}{2\pi r} \, dr = \frac{1}{2\pi}(\log(\varepsilon^s) - \log \varepsilon) = \frac{1}{2\pi}(1-s)|\log \varepsilon| = (1-s)E_\varepsilon^{\text{self}}.$$

As we will see, in view of Korn inequality, the logarithmic behavior and the concentration phenomenon of the self-energy hold true also in the case of elastic energies depending only on β^{sym} like in (7). In this respect the position of the dislocation and the shape of Ω itself do not have a big impact on the value of the self-energy. It seems then convenient to introduce the self-energy as a quantity depending only on the Burgers vector b (and the elasticity tensor $\mathbb{C}$) through a cell-problem. Before doing that, we proceed heuristically, by introducing the notion of *hard core* of the dislocation δ_0, as a region surrounding the dislocation such that

  i) The hard core region contains almost all the self-energy;
  ii) The hard core region shrinks at the dislocation point as the atomic scale ε tends to zero.

To this aim, in view of the previous discussion, it is enough to define the core region as $B_{\rho_\varepsilon}(0)$, where the hard core radius ρ_ε satisfies

  i) $\lim_{\varepsilon \to 0} \frac{\rho_\varepsilon}{\varepsilon^s} = \infty$ for every fixed $0 < s < 1$;
  ii) $\rho_\varepsilon^2 \to 0$ as $\varepsilon \to 0$.

Note that condition $i)$ is indeed equivalent to

  i') $\lim_{\varepsilon \to 0} \frac{\log(\rho_\varepsilon)}{\log \varepsilon} = 0$.

A direct consequence of our definitions is that the self-energy can be identified (up to lower order terms) with the elastic energy stored in the hard core region, for which, with a little abuse, we will use the same notation, i.e.,

$$(11) \qquad E_\varepsilon^{\text{self}} := \int_{B_{\rho_\varepsilon}(0)} |\beta(\mu)|^2 \, dx.$$

Consider next the case of a generic configuration of dislocations $\mu_\varepsilon := \sum_{i=1}^{N_\varepsilon} b \, \delta_{x_i}$ in Ω (but for sake of simplicity we will keep the Burgers vector equal to b and the toy elastic energy (9)). The hard core region of μ_ε is given by the union of the balls $B_{\rho_\varepsilon}(x_i)$.

Again we require a decay for ρ_ε and that the area of the hard core region tends to zero as $\varepsilon \to 0$, $N_\varepsilon \to \infty$, i.e.,

i) $\lim_{\varepsilon \to 0} \frac{\rho_\varepsilon}{\varepsilon^s} = \infty$ for every fixed $0 < s < 1$;

ii) $N_\varepsilon \rho_\varepsilon^2 \to 0$ as $\varepsilon \to 0$.

Moreover, we require that the dislocations are separated by a distance $2\rho_\varepsilon$, i.e.,

iii) the balls $B_{\rho_\varepsilon}(x_i)$ are pairwise disjoint.

This condition motivates the name of the hard core region. In view of this assumption, it is natural to identify the self-energy of μ_ε as the elastic energy stored in the hard core region. Therefore, we set

$$E_\varepsilon^{\text{self}}(\mu_\varepsilon) := \int_{\text{Hard Core}} |\beta(\mu_\varepsilon)|^2 \, dx = \sum_{i=1}^{N_\varepsilon} \int_{B_{\rho_\varepsilon}(x_i)} |\beta(\mu_\varepsilon)|^2 \, dx.$$

We expect each term of the sum in the right hand side to be asymptotically equivalent to the self-energy of a single dislocation introduced in (11), so that in view of (10),

$$(12) \qquad E_\varepsilon^{\text{self}}(\mu_\varepsilon) = \sum_{i=1}^{N_\varepsilon} \int_{B_{\rho_\varepsilon}(x_i)} |\beta(\mu_\varepsilon)|^2 \, dx \approx N_\varepsilon |\log \varepsilon|.$$

This expression represents the asymptotic behavior of the self-energy as $\varepsilon \to 0$, $N_\varepsilon \to \infty$.

2.2. **Long range interaction between dislocations.** Since the self-energy is concentrated in a small region surrounding the dislocations, it is natural to define the *interaction energy* $E_\varepsilon^{\text{inter}}$ as the energy diffused in the remaining part of the domain, far from each dislocation. In view of this definition we have that the total energy $E_\varepsilon(\mu_\varepsilon, \beta(\mu_\varepsilon))$ is given by the sum of the self and the interaction energy:

$$E_\varepsilon(\mu_\varepsilon, \beta(\mu_\varepsilon)) = \int_{\Omega_\varepsilon(\mu_\varepsilon)} W(\beta(\mu_\varepsilon)) \, dx = \int_{\text{Hard Core}} W(\beta(\mu_\varepsilon) \, dx$$

$$+ \int_{\Omega \setminus \text{Hard Core}} W(\beta(\mu_\varepsilon)) \, dx = E_\varepsilon^{\text{self}}(\mu_\varepsilon) + E_\varepsilon^{\text{inter}}(\mu_\varepsilon),$$

where we recall that $\Omega_\varepsilon(\mu_\varepsilon)$ is the region of Ω outside the dislocation cores, i.e., $\Omega_\varepsilon(\mu_\varepsilon) := \Omega \setminus \cup_{i=1}^{N_\varepsilon} B_\varepsilon(x_i)$, for $\mu_\varepsilon = \sum_{i=1}^{N_\varepsilon} \delta_{x_i} b$. Let us compute heuristically the interaction energy for a very simple configuration of dislocations and for the toy energy considered in (9). Let Ω be the unit ball B_1, and μ_ε be a configuration of N_ε periodically distributed dislocations whose Burgers vector b has modulus one. The stored interaction energy is given by

$$E_\varepsilon^{\text{inter}}(\mu_\varepsilon, \beta(\mu_\varepsilon)) := \int_{\Omega \setminus \text{Hard Core}} |\beta(\mu_\varepsilon)|^2 \, dx = \int_0^1 dr \int_0^{2\pi r} \chi_\varepsilon |\beta(r, \theta)|^2 \, d\theta,$$

where χ_ε denotes the characteristic function of the set $\Omega \setminus \text{Hard Core}$. Thanks to the uniform distribution of the x_i's in $B_1 = \Omega$ we deduce that the number of dislocations contained in each ball of radius r is proportional to the area of such a ball, and more precisely is of order $\pi r^2 N_\varepsilon$. Therefore, the average $\overline{\beta}_t(r)$ of the tangential component of the strain on each circle ∂B_r outside the dislocation hard cores, i.e., of $\beta_t(r, \theta) \chi_\varepsilon$, is of order $\pi r^2 N_\varepsilon / 2\pi r = N_\varepsilon r / 2$. The error is small thanks to the fact that the area of the hard cores region is negligible. By Jensen's inequality we obtain the following estimate from above for the interaction energy

$$E_\varepsilon^{\text{inter}}(\mu_\varepsilon) = \int_0^1 \int_0^{2\pi r} \chi_\varepsilon |\beta(r, \theta)|^2 \, d\theta \, dr \geq \int_0^1 2\pi r |\overline{\beta}_t(r)|^2 dr \approx \int_0^1 2\pi N_\varepsilon^2 r^3 / 4 dr = C N_\varepsilon^2,$$

where C is a constant independent of ε.

Note that the self-energy has been estimated looking at the circulation condition that the strain $\beta_\varepsilon(\mu)$ has to satisfy (in order to be an admissible strain) on the circles $\partial B_r(x_i)$ for $\varepsilon \leq r \leq \rho_\varepsilon$, while the estimate for the interaction energy has been established looking at the circulation condition on all circles ∂B_r for $r \leq 1$. We will see that this estimate is indeed sharp for recovery sequences in Γ-convergence. We will then conclude that the interaction energy behaves like N_ε^2 as $\varepsilon \to 0$, $N_\varepsilon \to \infty$.

2.3. Energy regimes. In view of the heuristic arguments of the previous section we fix a function $\varepsilon \mapsto N_\varepsilon$ that represents the number of dislocations present in the crystal corresponding to the internal scale ε. The above considerations can be summarized in three regimes for the behavior of N_ε with respect to $\varepsilon \to 0$:

1. Dilute dislocations ($N_\varepsilon \ll |\log \varepsilon|$): in this regime we have that the self-energy, which is of order $N_\varepsilon |\log \varepsilon|$, is predominant with respect to the interaction energy.
2. Critical regime ($N_\varepsilon \approx |\log \varepsilon|$): in this regime we have that the self-energy and the interaction energy are both of order $N_\varepsilon |\log \varepsilon| \approx |\log \varepsilon|^2$.
3. Super-critical regime ($N_\varepsilon \gg |\log \varepsilon|$): in this regime the interaction energy, which is of order $|N_\varepsilon|^2$, is predominant with respect to the self-energy.

3. THE SETTING OF THE PROBLEM

In this section we specify the mathematical setting of the problem. In particular, we introduce the class X_ε of admissible configurations of dislocations, the class $\mathcal{AS}_\varepsilon(\mu)$ of the corresponding admissible strains, and the rescaled energy functionals $\mathcal{F}_\varepsilon$.

From now on Ω is a bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary, representing a horizontal section of an infinite cylindrical crystal. For the given crystal, we introduce the class of Burgers vectors $S = \{b_1, ..., b_s\}$. In what follows we assume that S contains at least two (independent) vectors, so that

$$(13) \qquad\qquad \mathrm{Span}_\mathbb{R} S = \mathbb{R}^2$$

and this will imply that the function φ in the energy (8) is finite in whole of $\mathbb{R}^2$. The case of only one Burgers vector is easier and it implies that φ is finite only on a one dimensional subspace of $\mathbb{R}^2$. We denote by $\mathbb{S}$ the span of S with integer coefficients ($\mathbb{S} = \mathrm{Span}_\mathbb{Z} S$), i.e., the set of Burgers vectors for "multiple dislocations".

As in Section 2, N_ε represents the number of dislocations present in the crystal, corresponding to the internal scale ε. We introduce also the sequence ρ_ε representing the radius of the hard core surrounding the dislocations, and we require

   i) $\lim_{\varepsilon \to 0} \frac{\rho_\varepsilon}{\varepsilon_n^s} = \infty$ for every fixed $0 < s < 1$;

   ii) $\lim_{\varepsilon \to 0} |N_\varepsilon| \rho_\varepsilon^2 = 0$.

Condition i) says that the hard core region contains almost all the self-energy, while condition ii) says that the area of the hard core region tends to zero, and hence that its complement contains almost all the interaction energy. Note that conditions i) and ii) are compatible whenever

$$(14) \qquad\qquad N_\varepsilon \varepsilon^s \to 0 \qquad \text{for every fixed } s > 0.$$

We assume that the distance between any pair of dislocation points is at least $2\rho_\varepsilon$ and we define the class X_ε of admissible dislocations by

$$(15) \quad X_\varepsilon := \left\{ \mu \in \mathcal{M}(\Omega; \mathbb{R}^2) : \mu = \sum_{i=1}^{M} \xi_i \, \delta_{x_i}, \ M \in \mathbb{N}, \ B_{\rho_\varepsilon}(x_i) \subset \Omega, \right.$$

$$\left. |x_j - x_k| \geq 2\rho_\varepsilon \text{ for every } j \neq k, \xi_i \in \mathbb{S} \right\},$$

where $\mathcal{M}(\Omega; \mathbb{R}^2)$ denotes the space of vector valued Radon measures. Given $\mu \in X_\varepsilon$ and $r \in \mathbb{R}$, we define

$$(16) \qquad\qquad \Omega_r(\mu) := \Omega \setminus \bigcup_{x_i \in \mathrm{supp}(\mu)} \overline{B}_r(x_i),$$

where $B_r(x_i)$ denotes the open ball of center x_i and radius r.

The class of *admissible strains* associated with any $\mu := \sum_{i=1}^{M} \xi_i \, \delta_{x_i} \in X_\varepsilon$ is then

$$\left\{ \beta \in L^2(\Omega_\varepsilon(\mu); \mathbb{M}^{2\times 2}) : \mathrm{Curl}\, \beta = 0 \text{ in } \Omega_\varepsilon(\mu) \text{ and } \int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \xi_i \text{ for all } i = 1, ..., M \right\}.$$

Here t denotes the tangent to $\partial B_\varepsilon(x_i)$ and the integrand $\beta \cdot t$ is intended in the sense of traces (see Theorem 2, page 204 in [8] and Remark 1 for more details). In the arguments below it will be useful to extend the admissible strains to the whole of Ω. There are various extensions that can be considered and that are compatible with the model that we have in mind. We decide to extend the β's setting their value to be zero in the dislocation cores. Thus, from now on the class $\mathcal{AS}_\varepsilon(\mu)$ of *admissible strains* associated with any $\mu := \sum_{i=1}^M \xi_i \, \delta_{x_i}$ is given by

$$(17) \qquad \mathcal{AS}_\varepsilon(\mu) := \Big\{ \beta \in L^2(\Omega; \mathbb{M}^{2\times 2}) : \beta \equiv 0 \text{ in } \Omega \setminus \Omega_\varepsilon(\mu), \operatorname{Curl} \beta = 0 \text{ in } \Omega_\varepsilon(\mu),$$

$$\int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \xi_i, \text{ and } \int_{\Omega_\varepsilon(\mu)} (\beta - \beta^T) \, dx = 0 \text{ for all } i = 1, ..., M \Big\}.$$

In view of the definition of the elastic energy, the last condition in the definition of $\mathcal{AS}_\varepsilon(\mu)$ is not restrictive and it is there to guarantee uniqueness of the minimizing strain.

Remark 1. Let $\beta \in \mathcal{AS}_\varepsilon(\mu)$. Formally $\operatorname{Curl} \beta$ is a tensor defined by

$$(\operatorname{Curl} \beta)_{ijl} := \frac{\partial}{\partial x_l} \beta_{ij} - \frac{\partial}{\partial x_j} \beta_{il},$$

which by definition is antisymmetric with respect to the entries j, l. Therefore $\operatorname{curl} \beta_{(i)}$ (where $\beta_{(i)}$ denotes the ith row of β) can be identified with the scalar

$$\operatorname{curl} \beta_{(i)} := \frac{\partial}{\partial x_1} \beta_{i\,2} - \frac{\partial}{\partial x_2} \beta_{i\,1}.$$

In the sense of distribution, $\operatorname{curl} \beta_{(i)}$ is given by

$$(18) \qquad\qquad < \operatorname{curl} \beta_{(i)}, \varphi > = \int_\Omega < \beta_{(i)}, J\nabla\varphi >,$$

where J is the clockwise rotation of 90°. From equation (18) it turns out that whenever $\beta_{(i)}$ is in $L^2(\Omega; \mathbb{R}^2)$, then $\operatorname{curl} \beta_{(i)}$ is well defined for $\varphi \in H^1(\Omega)$ and acts continuously on it; so that

$$\operatorname{curl} \beta_{(i)} \in H^{-1}(\Omega; \mathbb{R}^2) \qquad \text{for every } \beta \in \mathcal{AS}_\varepsilon(\mu).$$

On the other hand, if $\beta_{(i)} \in L^1(\Omega; \mathbb{R}^2)$ and $\operatorname{curl} \beta_{(i)} \in H^{-1}(\Omega; \mathbb{R}^2)$ then $\beta_{(i)}$ is in $L^2(\Omega; \mathbb{R}^2)$ modulo gradients.

Finally, notice that for every $\mu := \sum_{i=1}^M \xi_i \, \delta_{x_i} \in X_\varepsilon$, the circulation condition in (17) can be stated in the following equivalent way:

$$< \operatorname{Curl} \beta, \varphi > = \sum_{i=1}^M \xi_i \, c_i,$$

for every $\varphi \in H_0^1(\Omega)$ such that $\varphi \equiv c_i$ in $B_\varepsilon(x_i)$. In particular, if φ belongs also to $C_0^0(\Omega)$, we have

$$< \operatorname{Curl} \beta, \varphi > = \int_\Omega \varphi \, d\mu.$$

The elastic energy E_ε corresponding to a pair (μ, β), with $\mu \in X_\varepsilon$ and $\beta \in \mathcal{AS}_\varepsilon(\mu)$, is defined by

$$(19) \qquad\qquad E_\varepsilon(\mu, \beta) := \int_{\Omega_\varepsilon(\mu)} W(\beta) \, dx,$$

where

$$W(\xi) := \frac{1}{2} \mathbb{C}\xi : \xi$$

is the *strain energy density*, and the elasticity tensor $\mathbb{C}$ satisfies

$$(20) \qquad\qquad c_1 |\xi^{\mathrm{sym}}|^2 \leq \mathbb{C}\xi : \xi \leq c_2 |\xi^{\mathrm{sym}}|^2 \qquad \text{for any } \xi \in \mathbb{M}^{2\times 2},$$

where $\xi^{\mathrm{sym}} := \frac{1}{2}(\xi + \xi^\top)$ and c_1 and c_2 are two given positive constants. Since β is always extended to zero outside $\Omega_\varepsilon(\mu)$, we can rewrite the energy as

$$E_\varepsilon(\mu, \beta) = \int_\Omega W(\beta) \, dx.$$

Note that for any given $\mu \in X_\varepsilon$, the problem

$$\min_{\beta \in \mathcal{AS}_\varepsilon(\mu)} \int_{\Omega_\varepsilon(\mu)} W(\beta)\, dx$$

has a unique solution. The proof of this fact can be obtained by taking of a finite number of segments by $\Omega_\varepsilon(\mu)$, obtaining in such a way a simply connected domain where to apply Korn inequality to the curl-free admissible strains, and then following the direct method of calculus of variations.

We conclude giving the notation of the rescaled functionals in the different regimes for N_ε, according to the asymptotic analysis given in the previous section:

Sub-critical or dilute regime: $N_\varepsilon \ll |\log \varepsilon|$. In this case the predominant contribution comes from the self-energy and is of order $N_\varepsilon |\log \varepsilon|$. Therefore, we set

(21) $$\mathcal{F}_\varepsilon^{\text{dilute}}(\mu, \beta) := \begin{cases} \frac{1}{N_\varepsilon |\log \varepsilon|} E_\varepsilon(\mu, \beta) & \text{if } \mu \in X_\varepsilon,\ \beta \in \mathcal{AS}_\varepsilon(\mu), \\ \infty & \text{otherwise in } L^2(\Omega; \mathbb{M}^{2\times 2}). \end{cases}$$

Critical regime: $N_\varepsilon = |\log \varepsilon|$. In this case both energies are of order $N_\varepsilon |\log \varepsilon| = |\log \varepsilon|^2$. Therefore, we set

(22) $$\mathcal{F}_\varepsilon(\mu, \beta) := \begin{cases} \frac{1}{|\log \varepsilon|^2} E_\varepsilon(\mu, \beta) & \text{if } \mu \in X_\varepsilon,\ \beta \in \mathcal{AS}_\varepsilon(\mu), \\ \infty & \text{otherwise in } L^2(\Omega; \mathbb{M}^{2\times 2}) \end{cases}$$

Super-critical regime: $N_\varepsilon \gg |\log \varepsilon|$. In this case the interaction energy is the leading term, and is of order N_ε^2. Therefore, we set

(23) $$\mathcal{F}_\varepsilon^{\text{super}}(\mu, \beta) := \begin{cases} \frac{1}{N_\varepsilon^2} E_\varepsilon(\mu, \beta) & \text{if } \mu \in X_\varepsilon,\ \beta \in \mathcal{AS}_\varepsilon(\mu), \\ \infty & \text{otherwise in } L^2(\Omega; \mathbb{M}^{2\times 2}) \end{cases}$$

Remark 2. The choice of representing the Curl constraint by defining the measure μ as a sum of Dirac masses concentrated in the dislocation points is one of the possible choices. Other possibilities would be to consider more regular measures with the same mass, such as

(24) $$\tilde{\mu} = \sum_{i=1}^{M} \frac{\chi_{B_\varepsilon(x_i)}}{\pi \varepsilon^2} \xi(x_i)$$

or

(25) $$\hat{\mu} = \sum_{i=1}^{M} \frac{\mathcal{H}^1 \llcorner \partial B_\varepsilon(x_i)}{2\pi\varepsilon} \xi(x_i)\,.$$

The advantage of these alternative choices is that $\tilde{\mu}$ and $\hat{\mu}$ belong to $H^{-1}(\Omega; \mathbb{R}^2)$, which is the natural space for $\text{Curl}\,\beta$. Indeed, since $\tilde{\mu}$ ($\hat{\mu}$ respectively) are in $H^{-1}(\Omega; \mathbb{R}^2)$, we could rewrite the class of admissible strains as follows:

$$\left\{ \beta \in L^1(\Omega; \mathbb{M}^{2\times 2}) : \text{Curl}\,\beta = \tilde{\mu} \text{ (respectively } \hat{\mu} \text{) in } \Omega \right\}.$$

This notion of admissible strains does not coincide with that given in (17), but it turns out to be equivalent to (17), in terms of Γ-convergence, in the study of the asymptotic behavior of the energy functionals as $\varepsilon \to 0$.

4. Cell formula for the self-energy

The self-energy stored in a neighborhood of a dislocation is of order $|\log \varepsilon|$ and, in view of the concentration of the energy, it is asymptotically not affected by the shape of the domain, depending only on the elasticity tensor $\mathbb{C}$ and on the Burgers vector b. It seems then natural to introduce rigorously the notion of self-energy through a cell problem.

In the following we will consider the self-energy of any multiple Burgers vector $\xi \in \mathbb{S}$. For convenience we will introduce all the quantities we need for a generic vector $\xi \in \mathbb{R}^2$.

For every $\xi \in \mathbb{R}^2$ and for every $0 < r_1 < r_2 \in \mathbb{R}$, let

(26) $$\mathcal{AS}_{r_1, r_2}(\xi) := \left\{ \beta \in L^2(B_{r_2} \setminus B_{r_1}; \mathbb{M}^{2\times 2}) : \text{Curl}\,\beta = 0,\ \int_{\partial B_{r_1}} \beta \cdot t\, ds = \xi \right\},$$

where B_r denotes the ball of radius r and center 0.

We first note that the admissibility conditions above on a strain β, assure an a priori bound from below for its energy. This is made precise by the following remark.

Remark 3. Given $0 < r_1 < r_2$ and $\xi \in \mathbb{R}^2$, for every admissible configuration $\beta \in \mathcal{AS}_{r_1,r_2}(\xi)$ we have

$$\int_{B_{r_2} \setminus B_{r_1}} |\beta^{\mathrm{sym}}|^2 \, dx \geq c|\xi|^2 ,$$

where the constant c depends only on r_1 and r_2.

Indeed, by introducing a cut with a segment L, the set $(B_{r_2} \setminus B_{r_1}) \setminus L$ becomes simply connected. Since $\mathrm{Curl}\, \beta = 0$ in $(B_{r_2} \setminus B_{r_1}) \setminus L$, there exists a function $u \in H^1(B_{r_2} \setminus B_{r_1}; \mathbb{R}^2)$ such that $\nabla u = \beta$ in $(B_{r_2} \setminus B_{r_1}) \setminus L$. By the classical Korn's inequality applied to u we obtain

$$\int_{B_{r_2} \setminus B_{r_1}} |\beta - A|^2 \, dx \leq C \int_{B_{r_2} \setminus B_{r_1}} |\beta^{\mathrm{sym}}|^2 \, dx,$$

for some skew symmetric matrices A. Moreover, by the fact that $\beta \in \mathcal{AS}_{r_1,r_2}(\eta_n)$, we conclude

$$\begin{aligned}
\int_{B_{r_2} \setminus B_{r_1}} |\beta - A|^2 \, dx &\geq \int_{r_1}^{r_2} \frac{1}{2\pi\rho} \left| \int_{\partial B_\rho} (\beta - A) \cdot t \, ds \right|^2 d\rho \\
&= \int_{r_1}^{r_2} \frac{1}{2\pi\rho} \left| \int_{\partial B_\rho} \beta \cdot t \, ds \right|^2 d\rho = \int_{r_1}^{r_2} \frac{1}{2\pi\rho} |\xi|^2 d\rho = \frac{|\xi|^2 \log \frac{r_2}{r_1}}{2\pi} .
\end{aligned}$$

Set now $C_\varepsilon := B_1 \setminus B_\varepsilon$, and let $\psi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ be the function defined through the following cell problem

$$(27) \qquad \psi_\varepsilon(\xi) := \frac{1}{|\log \varepsilon|} \min_{\beta \in \mathcal{AS}_{\varepsilon,1}(\xi)} \int_{C_\varepsilon} W(\beta) \, dx \qquad \text{for every } \xi \in \mathbb{R}^2.$$

By (20), it is easy to see that problem (27) has a solution. We will denote by $\beta_\varepsilon(\xi)$ the (unique) solution of the cell problem (27) whose average is a symmetric matrix. Note that $\beta_\varepsilon(\xi)$ satisfies the boundary value problem

$$(28) \qquad \begin{cases} \mathrm{Div}\, \mathbb{C}\beta_\varepsilon(\xi) &= 0 \quad \text{in } C_\varepsilon, \\ \mathbb{C}\beta_\varepsilon(\xi) \cdot \nu &= 0 \quad \text{on } \partial C_\varepsilon. \end{cases}$$

Moreover, we will denote by $\beta_{\mathbb{R}^2}(\xi) : \mathbb{R}^2 \to \mathbb{R}^2$ the planar strain defined in all $\mathbb{R}^2$ corresponding to the dislocation centered in 0 with Burgers vector ξ. The strain $\beta_{\mathbb{R}^2}(\xi)$ is of the type

$$(29) \qquad \beta_{\mathbb{R}^2}(\xi)(r, \theta) = \frac{1}{r} \Gamma_\xi(\theta),$$

where the function Γ depends on the elastic properties of the crystal, namely on the elasticity tensor $\mathbb{C}$, and $\beta_{\mathbb{R}^2}(\xi)$ is a solution of the equation (we refer the reader to [2] for a detailed treatment of the subject)

$$(30) \qquad \begin{cases} \mathrm{Curl}\, \beta_{\mathbb{R}^2}(\xi) &= \xi\delta_0 \quad \text{in } \mathbb{R}^2, \\ \mathrm{Div}\, \mathbb{C}\beta_{\mathbb{R}^2}(\xi) &= 0 \quad \text{in } \mathbb{R}^2. \end{cases}$$

In the next lemma we will see that in the cell formula (27) we can assign a suitable boundary condition without affecting the asymptotic behavior of the energy $\psi_\varepsilon(\xi)$.

Lemma 4. *Let $\xi \in \mathbb{R}^2$ be fixed and let $\hat{\beta} \in \mathcal{AS}_{1,\varepsilon}(\xi)$. Assume that $\hat{\beta}$ satisfies*

$$(31) \qquad |\hat{\beta}(x)| \leq \frac{K|\xi|}{|x|} \qquad \text{for every } x \in C_\varepsilon \text{ and for some } K > 0.$$

Then there exists a function $\tilde{\beta} \in \mathcal{AS}_{1,\varepsilon}(\xi)$ such that

    1) *$\tilde{\beta}$ coincides with $\hat{\beta}$ in a neighborhood of ∂C_ε;*

    2) *$\int_{C_\varepsilon} W(\tilde{\beta}) \, dx \leq \int_{C_\varepsilon} W(\beta_\varepsilon(\xi)) \, dx + C|\xi|^2$, where $C > 0$ depends only on K.*

Proof. As in Remark 3 we make the annulus C_ε a simply connected domain by introducing a cut with a segment L and we denote it by $\tilde{C}_\varepsilon := C_\varepsilon \setminus L$. Since both $\hat{\beta}$ and $\beta_\varepsilon(\xi)$ are curl free in $\tilde{C}_\varepsilon$, we have

$$\beta_\varepsilon(\xi) = \nabla u, \qquad \hat{\beta} = \nabla v \qquad \text{for some } u, v \in H^1(\tilde{C}_\varepsilon; \mathbb{R}^2).$$

We want to construct $\tilde{\beta}$ as the gradient of a convex combination of u and v. To this purpose, let $\varphi_1 : (\varepsilon, 1/3) \to \mathbb{R}$ be a piecewise affine function defined by

$$\varphi_1(r) := \begin{cases} 0 & \text{for } r \in (\varepsilon, 2\varepsilon), \\ 1 & \text{for } r \in (3\varepsilon, 1/3), \\ \varphi_1 \text{ is linear} & \text{in } (2\varepsilon, 3\varepsilon) \end{cases}$$

and let $\varphi_2 : [1/3, 1) \to \mathbb{R}$ be defined by

$$\varphi_2(r) := \begin{cases} 1 & \text{for } r = 1/3, \\ 0 & \text{for } r \in [2/3, 1), \\ \varphi_2 \text{ is linear} & \text{in } (1/3, 2/3). \end{cases}$$

Define

$$C_\varepsilon^1 := \{x \in C_\varepsilon : 2\varepsilon \le |x| \le 3\varepsilon\}, \qquad C_\varepsilon^2 := \{x \in C_\varepsilon : 1/3 \le |x| \le 2/3\}.$$

Finally, set

$$\tilde{\beta}(x) := \begin{cases} \nabla\big(\varphi_1(|x|)(u(x) - c_1(u)) + (1 - \varphi_1(|x|))(v(x) - c_1(v))\big) & \text{for } |x| \le 1/3, \\ \nabla\big(\varphi_2(|x|)(u(x) - c_2(u)) + (1 - \varphi_2(|x|))(v(x) - c_2(v))\big) & \text{for } |x| > 1/3, \end{cases}$$

where $c_i(f)$ denotes the mean value of the function f on C_ε^i. It is easy to check that by construction $\tilde{\beta}$ belongs to $\mathcal{AS}_{1,\varepsilon}(\xi)$ and coincides with $\hat{\beta}$ in a neighborhood of ∂C_ε. It remains to prove property 2).

By construction we have that $\tilde{\beta}$ coincides with $\beta_\varepsilon(\xi)$ in the region

$$C_\varepsilon^3 := \{x \in C_\varepsilon : 3\varepsilon \le |x| \le 1/3\}.$$

Therefore, to conclude the proof of property 2) it is enough to compute the energy stored in $C_\varepsilon \setminus C_\varepsilon^3$.

In view of (20), of the fact that $\|\nabla\varphi_1\|_\infty \le C/\varepsilon$, and of the following Poincaré inequality

$$\int_{(B_{3r} \setminus B_{2r}) \setminus L} |u - c(u)|^2 \, dx \le cr^2 \int_{(B_{3r} \setminus B_{2r}) \setminus L} |\nabla u|^2 \, dx,$$

where $r > 0$ and $c(u)$ stands for the average of u over the domain, it is easy to check that property 2) holds provided the following estimates are established

$$\text{(32)} \qquad \int_{C_\varepsilon \setminus C_\varepsilon^3} |\nabla u|^2 \, dx \le C|\xi|^2, \qquad \int_{C_\varepsilon \setminus C_\varepsilon^3} |\nabla v|^2 \, dx \le C|\xi|^2,$$

for some C independent of ε.

It remains to prove (32). By (31) it follows that

$$\text{(33)} \qquad \int_{C_\varepsilon \setminus C_\varepsilon^3} |\nabla v|^2 \, dx \le C|\xi|^2 \left(\int_\varepsilon^{2\varepsilon} \frac{1}{t} \, dt + \int_{2/3}^1 \frac{1}{t} \, dt \right) \le C|\xi|^2.$$

Concerning u, in view of (28) and (30), we have that ∇u can be written as

$$\text{(34)} \qquad \nabla u = \beta_{\mathbb{R}^2}(\xi) + \nabla h,$$

with h given by

$$\nabla h(x) = -\int_{\partial B_1(0) \cup \partial B_\varepsilon(0)} \nabla G(x, y)\, \mathbb{C}\beta_{\mathbb{R}^2}(\xi) \cdot \nu(y) \, dy \qquad \text{for } x \in \tilde{C}_\varepsilon.$$

Here $G(x, y)$ is the Green function corresponding to the elasticity tensor $\mathbb{C}$, satisfying the equation

$$-\text{Div}_x \mathbb{C}\nabla_x G(x, y) = \delta_y I \qquad \text{in } \mathbb{R}^2$$

for every fixed y. It is well-known (see [2]) that

(35) $$|\nabla G(x)| \leq C/|x| \quad \text{for every} \quad x \in \mathbb{R}^2\,,$$

for some constant $C > 0$. From (34) and (35) it follows as in (33) that

$$\int_{C_\varepsilon \setminus C_\varepsilon^3} |\nabla u|^2 dx \leq 2 \int_{C_\varepsilon \setminus C_\varepsilon^3} \left(|\beta_{\mathbb{R}^2}(\xi)|^2 + |\nabla h|^2 \right)\, dx \leq C|\xi|^2,$$

which concludes the proof of (32) and therefore of the lemma. $\square$

From Lemma 4, together with (29), we deduce the following corollary

Corollary 5. *There exists a constant $C > 0$ such that for every $\xi \in \mathbb{R}^2$,*

(36) $$\psi_\varepsilon(\xi) \leq \frac{1}{|\log \varepsilon|} \int_{C_\varepsilon} W(\beta_{\mathbb{R}^2}(\xi))\, dx \leq \psi_\varepsilon(\xi) + \frac{C|\xi|^2}{|\log \varepsilon|}.$$

In particular, as $\varepsilon \to 0$ the functions ψ_ε converge pointwise to the function $\psi : \mathbb{R}^2 \to \mathbb{R}$, defined by

(37) $$\psi(\xi) := \lim_{\varepsilon \to 0} \psi_\varepsilon(\xi) = \frac{1}{|\log \varepsilon|} \lim_{\varepsilon \to 0} \int_{C_\varepsilon} W(\beta_{\mathbb{R}^2}(\xi))\, dx = \int_{\partial B_1(0)} W(\Gamma_\xi(\theta))\, d\theta.$$

More precisely, we have

(38) $$|\psi_\varepsilon(\xi) - \psi(\xi)| \leq \frac{C|\xi|^2}{|\log \varepsilon|}.$$

By means of a simple change of variable $\varepsilon \to \rho$ we have that the self-energy is indeed concentrated in a ρ-neighborhood of the dislocation points whenever $|\log \rho| \ll |\log \varepsilon|$. The precise statement is given in the next proposition.

Proposition 6. *For every $\varepsilon > 0$ let $\rho_\varepsilon > 0$ be such that $\log \rho_\varepsilon / \log \varepsilon \to 0$ as $\varepsilon \to 0$. Let $\bar{\psi}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ be defined through the following minimum problem*

(39) $$\bar{\psi}_\varepsilon(\xi) := \frac{1}{|\log \varepsilon|} \min \left\{ \int_{B_{\rho_\varepsilon} \setminus B_\varepsilon} W(\beta)\, dx : \beta \in \mathcal{AS}_{\varepsilon, \rho_\varepsilon}(\xi) \right\}.$$

Then $\bar{\psi}_\varepsilon = \psi_\varepsilon(1 + o(\varepsilon))$, where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to ξ. In particular, $\bar{\psi}_\varepsilon$ converge pointwise as $\varepsilon \to 0$ to the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ given in Corollary 5.

Moreover, let $\hat{\beta} \in \mathcal{AS}_{\varepsilon, \rho_\varepsilon}(\xi)$, $\xi \in \mathbb{R}^2$, be such that

(40) $$|\hat{\beta}(x)| \leq K \frac{|\xi|^2}{|x|}$$

for some $K \in \mathbb{R}$, and let $\tilde{\psi}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ be defined through the following minimum problem

(41) $$\tilde{\psi}_\varepsilon(\xi) := \frac{1}{|\log \varepsilon|} \min \left\{ \int_{B_{\rho_\varepsilon} \setminus B_\varepsilon} W(\beta)\, dx : \beta \in \mathcal{AS}_{\varepsilon, \rho_\varepsilon}(\xi),\, \beta \cdot t = \hat{\beta} \cdot t \text{ on } \partial B_\varepsilon \cup \partial B_{\rho_\varepsilon} \right\}.$$

Then $\tilde{\psi}_\varepsilon = \psi_\varepsilon(1 + o(\varepsilon))$, where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to ξ. In particular, $\tilde{\psi}_\varepsilon$ converge pointwise as $\varepsilon \to 0$ to the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ given in Corollary 5.

Remark 7. The error $o(\varepsilon)$ appearing in the expression of $\bar{\psi}$ and $\tilde{\psi}$ in Proposition (6) can be estimated as follows $o(\varepsilon) \approx \log \rho_\varepsilon / \log \varepsilon$.

We are now in a position to define the density $\varphi : \mathbb{R}^2 \to [0, \infty)$ of the self-energy through the following relaxation procedure

(42) $$\varphi(\xi) := \inf \left\{ \sum_{k=1}^N \lambda_k \psi(\xi_k) \; : \; \sum_{k=1}^N \lambda_k \xi_k = \xi\,,\; N \in \mathbb{N},\; \lambda_k \geq 0\,,\; \xi_k \in \mathbb{S} \right\}.$$

It follows from the definition that the function φ is positively 1-homogeneous and convex. Moreover, since $\psi(\xi) \geq C|\xi|^2$ for some $C \geq 0$, (that can be checked by its very definition), the inf in (42) is indeed a minimum.

Remark 8. Note that if for every $z_1, \ldots, z_s \in \mathbb{Z}$ we have

$$(43) \qquad \psi\left(\sum_{i=1}^{s} z_i b_i\right) \geq \sum_{i=1}^{s} z_i \psi(b_i),$$

then in the relaxation procedure given in (42) we can replace S with $\mathbb{S}$. More precisely, the formula for φ reduces to

$$(44) \qquad \varphi(\xi) := \min\left\{\sum_{i=1}^{s} |\lambda_i| \psi(b_i) : \sum_{i=1}^{s} \lambda_i b_i = \xi, \ b_i \in S\right\}.$$

Actually, condition (43) can be viewed as a condition that the class of Burgers vectors b_i of the given crystal has to satisfy in order to contain all the dislocation's defects observed in the crystal. In other words, if a dislocation corresponding to a vector $b := \sum_{i=1}^{s} z_i b_i$ stores an energy smaller than that obtained by separating all its components b_i, then b itself has to be considered as a Burgers vector of the crystal. A rigorous mathematical definition of the class of Burgers vectors corresponding to a given crystal could be to consider the set $\{b_1, \ldots b_s\} \subset \mathbb{S}$ of all vectors satisfying $\psi(b_i) = \varphi(b_i)$, where $\mathbb{S}$ is the set of slips under which the crystal is invariant. The Burgers vectors defined in such a way would always satisfy property (43).

5. A KORN TYPE INEQUALITY FOR FIELDS WITH PRESCRIBED CURL

Let $u \in W^{1,2}(\Omega; \mathbb{R}^2)$, β its gradient and β^{sym} and β^{skew} its decomposition in symmetric and antisymmetric part. The classical Korn inequality asserts that if β^{skew} has zero mean value, then its L^2 norm is controlled by the L^2 norm of β^{sym}. We will show that in dimension two the same result holds true also for fields β that are not curl free, modulo an error depending actually on the mass of $\mathrm{Curl}\,\beta$. The result is a consequence of some estimates for elliptic systems with L^1-data recently proved by Bourgain, Brezis and Van Schaftingen in [3], [5]. The precise statement is the following

Theorem 9 (A Generalized Korn type inequality). *There exists a constant C depending only on Ω such that for every $\beta \in L^1(\Omega; \mathbb{M}^{2\times 2})$ with*

$$\mathrm{Curl}\,\beta = \mu \in \mathcal{M}(\Omega; \mathbb{R}^2), \qquad \int_\Omega \beta^{\mathrm{skew}} = 0,$$

we have

$$(45) \qquad \int_\Omega |\beta^{\mathrm{skew}}|^2\,dx \leq C\left(\int_\Omega |\beta^{\mathrm{sym}}|^2\,dx + \left(|\mu|(\Omega)\right)^2\right).$$

Proof. The condition $\mathrm{Curl}\,\beta = (\mu_1, \mu_2)$ can be written in the following form

$$\begin{cases} (\beta_{12}^{\mathrm{skew}})_{x_1} = & h_1 + \mu_1, \\ (\beta_{12}^{\mathrm{skew}})_{x_2} = & h_2 + \mu_2, \end{cases}$$

where $h_i \in H^{-1}(\Omega)$ are linear combinations of derivatives of entries of β^{sym}.

Since the field $\left((\beta_{12}^{\mathrm{skew}})_{x_1}, (\beta_{12}^{\mathrm{skew}})_{x_2}\right)$ is curl free, we deduce that $\mathrm{curl}\,(\mu_1, \mu_2) = \mathrm{curl}\,(-h_1, -h_2)$, or equivalently

$$(46) \qquad \mathrm{div}\,(-\mu_2, \mu_1) = \mathrm{div}\,(h_2, -h_1).$$

By [5, Lemma 3.3 and Remark 3.3] (see also [25] and [3]) we have that if $f \in L^1(\Omega; \mathbb{R}^2)$ is a vector field satisfying $\mathrm{div}\,f \in H^{-2}(\Omega)$, then f also belong to $H^{-1}(\Omega)$ and the following estimate also holds

$$\|f\|_{H^{-1}(\Omega)} \leq c\left(\|\mathrm{div}\,f\|_{H^{-2}(\Omega)} + \|f\|_{L^1(\Omega)}\right).$$

This result clearly extents by density to measures with bounded variations. Thus, by (46), we have that $\mathrm{div}\,(-\mu_2, \mu_1) \in H^{-2}(\Omega)$, and so we deduce that μ belongs to $H^{-1}(\Omega)$ and

$$(47) \qquad \|\mu\|_{H^{-1}(\Omega)} \leq c\left(\|\mathrm{div}\,(h_2, -h_1)\|_{H^{-2}(\Omega)} + |\mu|(\Omega)\right) \leq c\left(\|\beta^{\mathrm{sym}}\|_{L^2(\Omega; \mathbb{R}^2)} + |\mu|(\Omega)\right).$$

Now let $u \in H_0^1(\Omega; \mathbb{R}^2)$ be the solution of $-\Delta u = (-\mu_2, \mu_1)$ in Ω and let ξ be the 2×2 matrix defined by $\xi := J\nabla u$ (i.e., the ith row of ξ is given by $\xi_i = (-(u_i)_{x_2}, (u_i)_{x_1})$, for $i = 1, 2$). By definition we have that $\operatorname{Curl} \xi = \mu$. By (47) we then obtain

$$(48) \qquad \int_\Omega |\xi|^2 \, dx = \|\nabla u\|_{L^2}^2 \leq c\|\mu\|_{H^{-1}(\Omega)} \leq \left(\int_\Omega |\beta^{\mathrm{sym}}|^2 \, dx + \left(|\mu|(\Omega)\right)^2 \right).$$

Since the average of the anti-symmetric part of ξ can be easily estimated with the L^2 norm of ξ, we can assume that (48) holds for a matrix ξ such that ξ^{skew} has zero mean value and $\operatorname{Curl} \xi = \mu$. Therefore, by the classical Korn inequality applied to $\beta - \xi$, which by construction is curl free, and in view of (48) we conclude

$$\int_\Omega |\beta^{\mathrm{skew}}|^2 \, dx \leq c \left(\int_\Omega |\beta^{\mathrm{skew}} - \xi^{\mathrm{skew}}|^2 \, dx + \int_\Omega |\xi^{\mathrm{skew}}|^2 \, dx \right)$$

$$\leq c \left(\int_\Omega |\beta^{\mathrm{sym}} - \xi^{\mathrm{sym}}|^2 \, dx + \int_\Omega |\xi^{\mathrm{skew}}|^2 \, dx \right)$$

$$\leq c \left(\int_\Omega |\beta^{\mathrm{sym}}|^2 \, dx + \int_\Omega |\xi|^2 \, dx \right) \leq c \left(\int_\Omega |\beta^{\mathrm{sym}}|^2 \, dx + \left(|\mu|(\Omega)\right)^2 \right).$$

$$\square$$

6. THE CRITICAL REGIME ($N_\varepsilon \approx |\log \varepsilon|$)

In this section we will study the asymptotic behavior of the rescaled energy functionals as the internal scale $\varepsilon \to 0$, in the critical energy regime, namely with $N_\varepsilon = |\log \varepsilon|$. In terms of Γ-convergence, we consider the rescaled energy functionals $\mathcal{F}_\varepsilon : \mathcal{M}(\Omega; \mathbb{R}^2) \times L^1(\Omega; \mathbb{M}^{2 \times 2}) \to \mathbb{R}$ defined in (22).

According to the heuristic arguments above, in this regime we aspect the coexistence of the two effects, the interaction energy and the self-energy, so that the candidate for the Γ-limit $\mathcal{F} : \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2 \times 2}) \to \mathbb{R}$ is defined by

$$(49) \qquad \mathcal{F}(\mu, \beta) := \begin{cases} \displaystyle\int_\Omega W(\beta) \, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu| & \text{if } \mu \in H^{-1}(\Omega; \mathbb{R}^2), \operatorname{Curl} \beta = \mu; \\ \infty & \text{otherwise in } L^2(\Omega; \mathbb{M}^{2 \times 2}). \end{cases}$$

Theorem 10. *The following Γ-convergence result holds.*

   i) **Compactness.** *Let $\varepsilon_n \to 0$ and let $\{(\mu_n, \beta_n)\}$ be a sequence in $\mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2 \times 2})$ such that $\mathcal{F}_{\varepsilon_n}(\mu_n, \beta_n) \leq E$ for some positive constant E independent of n. Then there exist a subsequence of ε_n (not relabeled), a measure $\mu \in H^{-1}(\Omega; \mathbb{R}^2)$, and a strain $\beta \in L^2(\Omega; \mathbb{M}^{2 \times 2})$, with $\operatorname{Curl} \beta = \mu$, such that*

$$(50) \qquad \frac{1}{|\log \varepsilon_n|} \mu_n \overset{*}{\rightharpoonup} \mu \qquad \text{in } \mathcal{M}(\Omega; \mathbb{R}^2),$$

$$(51) \qquad \frac{1}{|\log \varepsilon_n|} \beta_n \rightharpoonup \beta \qquad \text{in } L^2(\Omega; \mathbb{M}^{2 \times 2}).$$

   ii) **Γ-convergence.** *The functionals $\mathcal{F}_\varepsilon$ Γ-converge to $\mathcal{F}$ as $\varepsilon \to 0$ with respect to the convergence in (50), (51), i.e., the following inequalities hold.*

      Γ-liminf inequality: *for every $(\mu, \beta) \in \left(\mathcal{M}(\Omega; \mathbb{R}^2) \cap H^{-1}(\Omega; \mathbb{R}^2)\right) \times L^2(\Omega; \mathbb{M}^{2 \times 2})$, with $\operatorname{Curl} \beta = \mu$, and for every sequence $(\mu_\varepsilon, \beta_\varepsilon) \in X_\varepsilon \times L^2(\Omega; \mathbb{M}^{2 \times 2})$ satisfying (50) and (51), we have*

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \geq \mathcal{F}(\mu, \beta).$$

      Γ-limsup inequality: *given $(\mu, \beta) \in \left(\mathcal{M}(\Omega; \mathbb{R}^2) \cap H^{-1}(\Omega; \mathbb{R}^2)\right) \times L^2(\Omega; \mathbb{M}^{2 \times 2})$, with $\operatorname{Curl} \beta = \mu$, there exists $(\mu_\varepsilon, \beta_\varepsilon) \in \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2 \times 2})$ satisfying (50) and (51), such that*

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \leq \mathcal{F}(\mu, \beta).$$

Remark 11. Consider the functionals $\mathcal{E}_\varepsilon(\mu) := \min_\beta \mathcal{F}_\varepsilon(\mu, \beta)$, representing the elastic energy induced by the dislocation measure μ. By the Γ-convergence result stated in Theorem 10 we immediately deduce that the functionals $\mathcal{E}_\varepsilon$ Γ-converge (as $\varepsilon \to 0$) to the functional $\mathcal{E} : \mathcal{M}(\Omega; \mathbb{R}^2) \cap H^{-1}(\Omega; \mathbb{R}^2) \to \mathbb{R}$ defined by

$$(52) \qquad \mathcal{E}(\mu) := \min_\beta \{ \mathcal{F}(\mu, \beta) : \operatorname{Curl} \beta = \mu \}.$$

Therefore, the energy $\mathcal{E}(\mu)$ induced by a distribution of dislocations μ in the critical regime is given by the sum of both an elastic and a plastic term. In particular, any distribution of dislocations in this regime induces a residual elastic distortion (i.e., if μ is not zero, then so is the corresponding strain β that minimizes (52) and hence its elastic energy).

6.1. **Compactness.** Let $\{(\mu_n, \beta_n)\}$ be a sequence in $\mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2\times 2})$ such that $\mathcal{F}_{\varepsilon_n}(\mu_n, \beta_n) \leq E$ for some positive constant E independent of n. We give the proof of the compactness property stated in Theorem 10 in three steps.

Step 1. *Weak compactness of the rescaled dislocation measures.*

We first show that the sequence $\{(1/|\log \varepsilon_n|)\mu_n\}$ is uniformly bounded in mass. Let $\mu_n = \sum_{i=1}^{M_n} \xi_{i,n} \delta_{x_{i,n}}$, with $\xi_{i,n} \in \mathbb{S}$; we claim that

$$(53) \qquad \frac{1}{|\log \varepsilon_n|}|\mu_n|(\Omega) = \frac{1}{|\log \varepsilon_n|} \sum_{i=1}^{M_n} |\xi_{i,n}| \leq C.$$

Indeed,

$$E \geq \mathcal{F}_{\varepsilon_n}(\mu_n, \beta_n) = \frac{1}{|\log \varepsilon_n|^2} \int_{\Omega_\varepsilon(\mu_n)} W(\beta_n) \, dx \geq \sum_{i=1}^{M_n} \frac{1}{|\log \varepsilon_n|^2} \int_{B_{\rho_{\varepsilon_n}}(x_{i,n})} W(\beta_n) \, dx \,,$$

where we recall that $\beta_n = 0$ in $\Omega \setminus \Omega_\varepsilon(\mu_n)$. After a change of variables we deduce

$$E \geq \sum_{i=1}^{M_n} \frac{1}{|\log \varepsilon_n|^2} \int_{B_{\rho_{\varepsilon_n}}} W(\beta_n(x_{i,n} + y)) \, dy.$$

Note that the functions $y \to \beta_n(x_{i,n} + y)$ belong to the class $\mathcal{AS}_{\varepsilon_n, \rho_{\varepsilon_n}}(\xi_{i,n})$ defined in (26). Therefore we have

$$(54) \quad E \geq \sum_{i=1}^{M_n} \frac{1}{|\log \varepsilon_n|^2} \int_{B_{\rho_{\varepsilon_n}}} W(\beta_n(x_i + y)) \, dy$$

$$\geq \frac{1}{|\log \varepsilon_n|} \sum_{i=1}^{M_n} \bar{\psi}_{\varepsilon_n}(\xi_{i,n}) = \frac{1}{|\log \varepsilon_n|} \sum_{i=1}^{M_n} |\xi_{i,n}|^2 \, \bar{\psi}_{\varepsilon_n}\left(\frac{\xi_{i,n}}{|\xi_{i,n}|}\right),$$

where the function $\bar{\psi}_{\varepsilon_n}$ is defined in (39). Let ψ be the function given by Corollary 5, and let $2c := \inf_{|\xi|=1} \psi(\xi)$. By Proposition 6 we deduce that for n large enough $\bar{\psi}_{\varepsilon_n}(\xi) \geq c$ for every ξ with $|\xi| = 1$. By (54) we obtain

$$E \geq \frac{1}{|\log \varepsilon_n|} \sum_{i=1}^{M_n} |\xi_{i,n}|^2 \bar{\psi}_{\varepsilon_n}\left(\frac{\xi_{i,n}}{|\xi_{i,n}|}\right) \geq \frac{c}{|\log \varepsilon_n|} \sum_{i=1}^{M_n} |\xi_{i,n}|^2 \geq \frac{C}{|\log \varepsilon_n|} \sum_{i=1}^{M_n} |\xi_{i,n}|,$$

where the last inequality follows from the fact that $\xi_{i,n} \in \mathbb{S} = \operatorname{Span}_{\mathbb{Z}} S$, S is a finite set, and hence $|\xi_{i,n}|$ are bounded away from zero. We conclude that (53) holds.

Step 2. *Weak compactness of the rescaled strains.*

In view of the coercivity condition (3) we have

$$(55) \qquad C|\log \varepsilon_n|^2 \geq C|\log \varepsilon_n|^2 \mathcal{F}_\varepsilon(\mu_n, \beta_n) \geq C \int_\Omega W(\beta_n) \, dx \geq \int_\Omega |\beta_n^{\mathrm{sym}}|^2 \, dx.$$

The idea of the proof is to apply the generalized Korn inequality provided by Theorem 9 to β_n to control the anti-symmetric part of β_n. Note that the Curl of β_n is clearly related to the dislocation measure μ_n, whose mass is bounded by $C|\log \varepsilon_n|$ by Step 1. On the other hand, it is not clear that

$|\mathrm{Curl}\,\beta_n| \le C|\log \varepsilon_n|$. Therefore we proceed as follows. For every $x_{i,n}$ in the support set of μ_n, set $C_{i,n} := B_{2\varepsilon_n}(x_{i,n}) \setminus B_{\varepsilon_n}(x_{i,n})$ and consider the function $K_{i,n} : C_{i,n} \to \mathbb{M}^{2\times 2}$ defined by

$$K_{i,n} := \frac{1}{2\pi} \xi_{i,n} \otimes J \frac{x - x_i}{|x - x_i|^2},$$

where J is the clockwise rotation of 90°. It easy to show that the following estimate holds true

$$\text{(56)} \qquad \int_{C_{i,n}} |K_{i,n}|^2 \, dx \le c \int_{C_{i,n}} |\beta_n^{\mathrm{sym}}|^2 \, dx,$$

Indeed, it is straightforward to check that

$$\int_{C_{i,n}} |K_{i,n}|^2 \, dx = C |\xi_{i,n}|^2$$

and by a scaling argument, from Remark 3 we get

$$\int_{C_{i,n}} |\beta_n^{\mathrm{sym}}|^2 \, dx \ge c |\xi_{i,n}|^2$$

and hence (56).

By construction $\mathrm{Curl}\,(\beta_n - K_{i,n}) = 0$ in $C_{i,n}$, and so $\beta_n - K_{i,n} = \nabla v_{i,n}$ in $C_{i,n}$ for some $v_{i,n} \in H^1(C_{i,n}; \mathbb{R}^2)$. Thus by (56) we have

$$\int_{C_{i,n}} |\nabla v_{i,n}^{\mathrm{sym}}|^2 \, dx \le C \int_{C_{i,n}} (|\beta_n^{\mathrm{sym}}|^2 + |K_{i,n}|^2) \, dx \le C \int_{C_{i,n}} |\beta_n^{\mathrm{sym}}|^2 \, dx \,,$$

and hence, applying the standard Korn's inequality to $v_{i,n}$, we deduce that

$$\int_{C_{i,n}} |\nabla v_{i,n} - A_{i,n}|^2 \, dx \le C \int_{C_{i,n}} |\nabla v_{i,n}^{\mathrm{sym}}|^2 \, dx \le C \int_{C_{i,n}} |\beta_n^{\mathrm{sym}}|^2 \, dx,$$

where $A_{i,n}$ is a suitable anti-symmetric matrix. By standard extension arguments there exists a function $u_{i,n} \in H^1(B_{2\varepsilon_n}(x_{i,n}); \mathbb{R}^2)$, such that $\nabla u_{i,n} \equiv \nabla v_{i,n} - A_{i,n}$ in $C_{i,n}$, and such that

$$\text{(57)} \qquad \int_{B_{2\varepsilon_n}(x_{i,n})} |\nabla u_{i,n}|^2 \, dx \le \int_{C_{i,n}} |\nabla v_{i,n} - A_{i,n}|^2 \, dx \le C \int_{C_{i,n}} |\beta_n^{\mathrm{sym}}|^2 \, dx.$$

Consider the field $\tilde{\beta}_n : \Omega \to \mathbb{M}^{2\times 2}$ defined by

$$\tilde{\beta}_n(x) := \begin{cases} \beta_n(x) & \text{if } x \in \Omega_{\varepsilon_n}, \\ \nabla u_{i,n}(x) + A_{i,n} & \text{if } x \in B_{\varepsilon_n}(x_{i,n}). \end{cases}$$

In view of (57) it follows

$$\int_{\Omega} |\tilde{\beta}_n^{\mathrm{sym}}|^2 \, dx = \int_{\Omega} |\beta_n^{\mathrm{sym}}|^2 \, dx + \sum_i \int_{B_{\varepsilon_n}(x_{i,n})} |\nabla u_{i,n}^{\mathrm{sym}}|^2 \, dx \le C \int_{\Omega} |\beta_n^{\mathrm{sym}}|^2 \, dx \le C |\log \varepsilon_n|^2.$$

By construction we have $|\mathrm{Curl}\,\tilde{\beta}_n|(\Omega) = |\mu_n|(\Omega)$; therefore, we can apply Theorem 9 to $\tilde{\beta}_n$, obtaining

$$\text{(58)} \qquad \int_{\Omega} |\tilde{\beta}_n - \tilde{A}_n|^2 \, dx \le C \left(\int_{\Omega} |\tilde{\beta}_n^{\mathrm{sym}}|^2 \, dx + \left(|\mu_n|(\Omega)\right)^2 \right) \le C |\log \varepsilon_n|^2,$$

where $\tilde{A}_n$ is the average of the anti-symmetric part of $\tilde{\beta}_n$. Since the average of β_n is a symmetric matrix, we have

$$\text{(59)} \qquad \int_{\Omega_{\varepsilon_n}(\mu_n)} |\beta_n|^2 \, dx = \int_{\Omega_{\varepsilon_n}(\mu_n)} |\beta_n - \tilde{A}_n|^2 - |\tilde{A}_n|^2 \, dx \le \int_{\Omega_{\varepsilon_n}(\mu_n)} |\beta_n - \tilde{A}_n|^2 \, dx.$$

Finally, by (58) and (59) we conclude

$$\int_{\Omega_{\varepsilon_n}(\mu_n)} |\beta_n|^2 \, dx \le \int_{\Omega_{\varepsilon_n}(\mu_n)} |\beta_n - \tilde{A}_n|^2 \, dx \le \int_{\Omega} |\tilde{\beta}_n - \tilde{A}_n|^2 \, dx \le C |\log \varepsilon_n|^2,$$

which gives the desired compactness property for $\beta_n / |\log \varepsilon_n|$ in $L^2(\Omega; \mathbb{M}^{2\times 2})$.

Step 3. μ belongs to $H^{-1}(\Omega; \mathbb{R}^2)$ and $\mathrm{Curl}\,\beta = \mu$.

Let $\varphi \in C_0^1(\Omega)$. It is easy to construct a sequence $\{\varphi_n\} \subset H_0^1(\Omega)$ converging to φ uniformly and strongly in $H_0^1(\Omega)$ and satisfying the property

$$\varphi_n \equiv \varphi(x_{i,n}) \qquad \text{in } B_{\varepsilon_n}(x_{i,n}) \quad \text{for every } x_{i,n} \text{ in the support set of } \mu_n.$$

By Remark 1 we have

$$
\begin{aligned}
\int_\Omega \varphi \, d\mu &= \lim_{n \to +\infty} \frac{1}{|\log \varepsilon_n|} \int_\Omega \varphi_n \, d\mu_n = \lim_{n \to +\infty} \frac{1}{|\log \varepsilon_n|} < \mathrm{Curl}\, \beta_n, \varphi_n > \\
&= \lim_{n \to +\infty} \frac{1}{|\log \varepsilon_n|} \int_\Omega \beta_n J \nabla \varphi_n \, dx = \int_\Omega \beta J \nabla \varphi \, dx = < \mathrm{Curl}\, \beta, \varphi >,
\end{aligned}
$$

from which we deduce the admissibility condition $\mathrm{Curl}\, \beta = \mu$. Moreover, since by the previous step we have $\beta \in L^2(\Omega; \mathbb{M}^{2\times 2})$, we deduce that μ belongs to $H^{-1}(\Omega; \mathbb{R}^2)$.

6.2. Γ-liminf inequality.
Here we prove the Γ-liminf inequality of Theorem 10. Let

$$(\mu, \beta) \in \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2\times 2}) \text{ with } \mathrm{Curl}\, \beta = \mu,$$

and let $(\mu_\varepsilon, \beta_\varepsilon)$ satisfy (50) and (51). In order to prove the Γ-liminf inequality

$$(60) \qquad \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \geq \mathcal{F}(\mu, \beta),$$

it is enough to show that inequality holds for the self and the interaction energy separately. More precisely, we write the energy corresponding to $(\mu_\varepsilon, \beta_\varepsilon)$ in the following way

$$\mathcal{F}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) = \int_\Omega \chi_{\cup B_{\rho_\varepsilon}(x_{i,\varepsilon})} W(\beta_\varepsilon) \, dx + \int_\Omega \chi_{\Omega \setminus \cup B_{\rho_\varepsilon}(x_{i,\varepsilon})} W(\beta_\varepsilon) \, dx$$

and we estimate the two terms separately. Set $\eta_\varepsilon := \sum_{i=1}^{M_\varepsilon} \delta_{x_{i,\varepsilon}}$. By Proposition 6 we have

$$(61) \qquad \frac{1}{|\log \varepsilon|} \int_\Omega \chi_{\cup B_{\rho_\varepsilon}(x_{i,\varepsilon})} W(\beta_n) \, dx \geq \int_\Omega \bar{\psi}_\varepsilon \left(\frac{d\mu_\varepsilon}{d\eta_\varepsilon} \right) d\eta_\varepsilon \geq \int_\Omega \psi \left(\frac{d\mu_\varepsilon}{d\eta_\varepsilon} \right) (1 + o(\varepsilon)) \, d\eta_\varepsilon,$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since $\mathrm{Span}_{\mathbb{R}} \mathbb{S} = \mathbb{R}^2$, the convex 1-homogeneous function φ defined in (42) is finite in $\mathbb{R}^2$, and so continuous. Thus, in view of Reshetnyak Theorem (see [22, Theorem 1.2]), we deduce

$$
\begin{aligned}
(62) \quad \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_\Omega \psi \left(\frac{d\mu_\varepsilon}{d\eta_\varepsilon} \right) d\eta_\varepsilon &\geq \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_\Omega \varphi \left(\frac{d\mu_\varepsilon}{d\eta_\varepsilon} \right) d\eta_\varepsilon \\
&= \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_\Omega \varphi \left(\frac{d\mu_\varepsilon}{d|\mu_\varepsilon|} \right) d|\mu_\varepsilon| \geq \int_\Omega \varphi \left(\frac{d\mu}{d|\mu|} \right) d|\mu|.
\end{aligned}
$$

From (61) and (62) we have that the Γ-liminf inequality holds for the self-energy, i.e.,

$$(63) \qquad \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \int_\Omega \chi_{\cup B_{\rho_\varepsilon}(x_{i,\varepsilon})} W(\beta_\varepsilon) \, dx \geq \int_\Omega \varphi \left(\frac{d\mu}{d|\mu|} \right) d|\mu|.$$

Concerning the interaction energy, by semicontinuity we immediately deduce

$$
\begin{aligned}
(64) \quad \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \int_\Omega \chi_{\Omega \setminus \cup B_{\rho_\varepsilon}(x_{i,\varepsilon})} W(\beta_\varepsilon) \, dx \\
= \liminf_{\varepsilon \to 0} \int_\Omega W \left(\chi_{\Omega \setminus \cup B_{\rho_\varepsilon}(x_{i,\varepsilon})} \frac{1}{|\log \varepsilon|} \beta_\varepsilon^{\mathrm{sym}} \right) dx \geq \int_\Omega W(\beta^{\mathrm{sym}}) \, dx.
\end{aligned}
$$

In the last inequality we used the fact that the number of atoms of μ_ε is bounded by $|\log \varepsilon|$ and hence, since by assumption $|\log \varepsilon| \rho_\varepsilon^2 \to 0$, the function $\chi_{\Omega \setminus \cup B_{\rho_\varepsilon}(x_{i,\varepsilon})}$ converges strongly to 1.

Inequality (63), together with inequality (64), gives the Γ-liminf inequality.

6.3. **Γ-limsup inequality.** Here we prove the Γ-limsup inequality of Theorem 10. We begin with a lemma that will be useful in the construction of recovery sequences also for different energy regimes under consideration. Given $\mu_\varepsilon := \sum_{i=1}^{M_\varepsilon} \xi_{i,\varepsilon} \delta_{x_{i,\varepsilon}} \in X_\varepsilon$ and $r_\varepsilon \to 0$, we introduce the corresponding measures, diffused on balls of radius r_ε and on circles of radius r_ε, respectively, defined by

$$(65) \qquad \tilde{\mu}_\varepsilon^{r_\varepsilon} := \sum_{i=1}^{M_\varepsilon} \frac{\chi_{B_{r_\varepsilon}(x_{i,\varepsilon})}}{\pi r_\varepsilon^2} \xi_{i,\varepsilon}, \qquad \hat{\mu}_\varepsilon^{r_\varepsilon} := \sum_{i=1}^{M_\varepsilon} \frac{\mathcal{H}^1 \llcorner \partial B_{r_\varepsilon}(x_{i,\varepsilon})}{2\pi r_\varepsilon} \xi_{i,\varepsilon} \,.$$

For every $x_{\varepsilon,i}$ in the support set of μ_ε we define the functions $\tilde{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}}$, $\hat{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}} : B_{r_\varepsilon}(x_{\varepsilon,i}) \to \mathbb{M}^{2\times 2}$ as follows

$$(66) \qquad \tilde{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}}(x) := \frac{1}{2\pi r_\varepsilon^2} \xi_{\varepsilon,i} \otimes J(x - x_{\varepsilon,i}), \qquad \hat{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}}(x) := \frac{1}{2\pi} \xi_{\varepsilon,i} \otimes J \frac{x - x_{\varepsilon,i}}{|x - x_{\varepsilon,i}|^2},$$

where J is the clockwise rotation of 90°. Finally, we introduce the functions $\tilde{K}_\varepsilon^{\mu_\varepsilon} : \Omega \to \mathbb{R}^2$, $\hat{K}_\varepsilon^{\mu_\varepsilon} : \Omega \to \mathbb{R}^2$, defined by

$$(67) \qquad \tilde{K}_\varepsilon^{\mu_\varepsilon} := \sum_{i=1}^{M_\varepsilon} \tilde{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}} \chi_{B_{r_\varepsilon}(x_{\varepsilon,i})}, \qquad \hat{K}_\varepsilon^{\mu_\varepsilon} := \sum_{i=1}^{M_\varepsilon} \hat{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}} \chi_{B_{r_\varepsilon}(x_{\varepsilon,i})} \,.$$

Note that

$$(68) \qquad \operatorname{Curl} \tilde{K}_\varepsilon^{\mu_\varepsilon} = \tilde{\mu}_\varepsilon^{r_\varepsilon} - \hat{\mu}_\varepsilon^{r_\varepsilon}, \qquad \operatorname{Curl} \hat{K}_\varepsilon^{\mu_\varepsilon} = -\hat{\mu}_\varepsilon^{r_\varepsilon} \,.$$

Lemma 12. *Let $N_\varepsilon \to \infty$ be satisfying (14), $\xi := \sum_{k=1}^{M} \lambda_k \xi_k$ with $\xi_k \in \mathbb{S}$, $\lambda_k \geq 0$, $\Lambda := \sum_k \lambda_k$, $\mu := \xi \, dx$ and $r_\varepsilon := 1/(2\sqrt{\Lambda N_\varepsilon})$. Then there exists a sequence of measures $\mu_\varepsilon = \sum_{k=1}^{M} \xi_k \mu_\varepsilon^k$ in X_ε, with μ_ε^k of the type $\sum_{l=1}^{M_\varepsilon^k} \delta_{x_{\varepsilon,l}}$ such that $B_{r_\varepsilon}(x) \subset \Omega$, $|x - y| \geq 2r_\varepsilon$ for every x, y in the support set of μ_ε, and such that*

$$(69) \qquad |\mu_\varepsilon^k|/N_\varepsilon \overset{*}{\rightharpoonup} \lambda_k \, dx \text{ in } \mathcal{M}(\Omega),$$

$$(70) \qquad \frac{\mu_\varepsilon}{N_\varepsilon} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega;\mathbb{R}^2), \qquad \frac{\hat{\mu}_\varepsilon^{r_\varepsilon}}{N_\varepsilon} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega;\mathbb{R}^2), \qquad \frac{\tilde{\mu}_\varepsilon^{r_\varepsilon}}{N_\varepsilon} \overset{*}{\rightharpoonup} \xi \text{ in } L^\infty(\Omega;\mathbb{R}^2),$$

$$(71) \qquad \frac{\tilde{\mu}_\varepsilon^{r_\varepsilon}}{N_\varepsilon} \to \mu, \qquad \frac{\tilde{\mu}_\varepsilon^{r_\varepsilon} - \hat{\mu}_\varepsilon^{r_\varepsilon}}{N_\varepsilon} \to 0 \qquad \text{strongly in } H^{-1}(\Omega;\mathbb{R}^2).$$

Proof. First, we prove the lemma for $M = 1$ and $\mu = \xi \, dx$ with $\xi \in \mathbb{S}$.

For $\mu := \xi \, dx$ we cover $\mathbb{R}^2$ with cubes of size $2r_\varepsilon$, we plug a mass with weight ξ in the center of all of such cubes which are contained in Ω, and we set μ_ε the measure obtained through this procedure. Let us prove (71), all the other properties following easily by the definition of μ_ε. Since $\mu - \frac{\tilde{\mu}_\varepsilon^{r_\varepsilon}}{N_\varepsilon}$ converges weakly to zero in $L^2(\Omega;\mathbb{R}^2)$, by the compact embedding of L^2 in H^{-1} we have that $\mu - \frac{\tilde{\mu}_\varepsilon^{r_\varepsilon}}{N_\varepsilon} \to 0$ in $H^{-1}(\Omega;\mathbb{R}^2)$. The fact that $\frac{\tilde{\mu}_\varepsilon^{r_\varepsilon} - \hat{\mu}_\varepsilon^{r_\varepsilon}}{N_\varepsilon} \to 0$ strongly in $H^{-1}(\Omega;\mathbb{R}^2)$ follows directly by (68), since $\tilde{K}_\varepsilon^{\mu_\varepsilon}/N_\varepsilon \to 0$ strongly in $L^2(\Omega;\mathbb{R}^2)$ (that can be checked by a very simple estimate).

The general case with $\xi \in \mathbb{R}^2$ and $M > 1$ will follow by approximating $\mu = \xi dx$ with periodic locally constant measures with weight ξ_k on sets with volume fraction λ_k/Λ. In each region where the approximating measure is constant we apply the construction above and then we take a diagonal sequence. $\qquad\square$

We are in a position to prove the Γ-limsup inequality of Theorem 10. We will proceed in several steps.

Step 1. *The case* $\mu \equiv \xi \, dx$.

Given $\xi \in \mathbb{R}^2$ and $\beta \in L^2(\Omega;\mathbb{M}^{2\times 2})$ with $\operatorname{Curl}\beta = \xi \, dx$, we will construct a recovery sequence $\{\mu_\varepsilon\} \subset X_\varepsilon$, $\beta_\varepsilon \in \mathcal{AS}_\varepsilon(\mu_\varepsilon)$, such that $(\mu_\varepsilon, \beta_\varepsilon)$ converges to $(\xi \, dx, \beta)$ in the sense of (50) and (51),

$$(72) \qquad \limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \int_\Omega W(\beta_\varepsilon) \, dx \leq \int_\Omega (W(\beta) + \varphi(\xi)) \, dx \,,$$

and satisfying the additional requirement that $(\beta_\varepsilon/|\log \varepsilon| - \beta) \cdot t$ tends to zero strongly in $H^{-\frac{1}{2}}(\partial\Omega)$.

Let φ be the self-energy density defined in (42), and let $\lambda_k \geq 0$, $\xi_k \in \mathbb{S}$ be such that $\xi = \sum_k \lambda_k \xi_k$ and

$$(73) \qquad \varphi(\xi) = \sum_{k=1}^{M} \lambda_k \psi(\xi_k).$$

Consider the sequence $\mu_\varepsilon := \sum_{i=1}^{M_\varepsilon} \xi_{\varepsilon,i} \delta_{x_{\varepsilon,i}}$ given by Lemma 12 with $N_\varepsilon = |\log \varepsilon|$. Note that, since $N_\varepsilon \rho_\varepsilon^2 \to 0$, we have that $r_\varepsilon \gg \rho_\varepsilon$, and so $\mu_\varepsilon \in X_\varepsilon$.

Since the function $\hat{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}}$ defined in (66) belongs to $\mathcal{AS}_{\varepsilon,\rho_\varepsilon}(\xi_{\varepsilon,i})$ and satisfies condition (40), by Proposition 6 for every $x_{\varepsilon,i}$ in the support set of μ_ε we can find a strain $\hat{\beta}_{\varepsilon,i} : \Omega \to \mathbb{M}^{2\times 2}$ such that

    1) $\hat{\beta}_{\varepsilon,i} \in \mathcal{AS}_{\varepsilon,\rho_\varepsilon}(\xi_{\varepsilon,i})$,
    2) $\hat{\beta}_{\varepsilon,i} \cdot t = \hat{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}} \cdot t$ on $\partial B_\varepsilon(x_{\varepsilon,i}) \cup \partial B_{\rho_\varepsilon}(x_{\varepsilon,i})$,
    3) $\frac{1}{|\log \varepsilon|} \int_{B_{\rho_\varepsilon}(x_{\varepsilon,i}) \setminus B_\varepsilon(x_{\varepsilon,i})} W(\hat{\beta}_{\varepsilon,i}) \, dx = \psi(\xi_{\varepsilon,i})(1 + o(\varepsilon))$ where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Extend $\hat{\beta}_{\varepsilon,i}$ to be $\hat{K}_{\varepsilon,i}^{\xi_{\varepsilon,i}}$ in $B_{r_\varepsilon}(x_{\varepsilon,i}) \setminus B_{\rho_\varepsilon}(x_{\varepsilon,i})$ and zero in $\Omega \setminus (B_{r_\varepsilon}(x_{\varepsilon,i}) \setminus B_\varepsilon(x_{\varepsilon,i}))$, and set

$$(74) \qquad \hat{\beta}_\varepsilon := \sum_{i=1}^{M_\varepsilon} \hat{\beta}_{\varepsilon,i}.$$

Then

    4) $\operatorname{Curl} \hat{\beta}_\varepsilon = -\hat{\mu}_\varepsilon^{r_\varepsilon} + \hat{\mu}_\varepsilon^\varepsilon$,

where $\hat{\mu}_\varepsilon^{r_\varepsilon}$ and $\hat{\mu}_\varepsilon^\varepsilon$ are defined in (65). Finally, set

$$(75) \qquad \bar{\beta}_\varepsilon := |\log \varepsilon| \beta - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \hat{\beta}_\varepsilon,$$

where $\tilde{K}_\varepsilon^{\mu_\varepsilon}$ is defined according to (67). By Lemma 12 and (68)

$$\operatorname{Curl} \frac{\bar{\beta}_\varepsilon}{|\log \varepsilon|} \llcorner \Omega_\varepsilon(\mu_\varepsilon) = (\mu - \frac{\tilde{\mu}_\varepsilon^{r_\varepsilon}}{|\log \varepsilon|} + \frac{\hat{\mu}_\varepsilon^{r_\varepsilon}}{|\log \varepsilon|} - \frac{\hat{\mu}_\varepsilon^{r_\varepsilon}}{|\log \varepsilon|}) = (\mu - \frac{\tilde{\mu}_\varepsilon^{r_\varepsilon}}{|\log \varepsilon|}) \to 0 \quad \text{in } H^{-1}(\Omega; \mathbb{R}^2).$$

Therefore, we can add to $\bar{\beta}_\varepsilon$ a vanishing sequence $\frac{R_\varepsilon}{|\log \varepsilon|} \to 0$ in $L^2(\Omega; \mathbb{M}^{2\times 2})$, obtaining the admissible strain

$$(76) \qquad \beta_\varepsilon := (\bar{\beta}_\varepsilon + R_\varepsilon) \chi_{\Omega_\varepsilon(\mu_\varepsilon)}.$$

In order to prove that the pair $(\mu_\varepsilon, \beta_\varepsilon)$ is the desired recovery sequence we have to check the following properties

    i) β_ε converge to β in the sense of definition (51);
    ii) The pair $(\beta_\varepsilon, \mu_\varepsilon)$ is a recovery sequence, i.e.,

$$\lim_{\varepsilon \to 0} \int_\Omega W(\beta_\varepsilon) \, dx = \int_\Omega (W(\beta) + \varphi(\xi)) \, dx.$$

To prove i), we first note that since $M_\varepsilon \sim |\log \varepsilon|$, $r_\varepsilon \sim 1/\sqrt{\varepsilon}$, we have that

$$\int_{\Omega_{\rho_\varepsilon}(\mu_\varepsilon)} \frac{|\hat{\beta}_\varepsilon|^2}{|\log \varepsilon|} dx = \int_{\Omega_{r_\varepsilon}(\mu_\varepsilon) \setminus \Omega_{\rho_\varepsilon}(\mu_\varepsilon)} \frac{|\hat{K}_\varepsilon^{\mu_\varepsilon}|^2}{|\log \varepsilon|} dx \to 0,$$

which implies that $\hat{\beta}_\varepsilon / |\log \varepsilon|$ is concentrated on the hard core region. Then by Lemma 12, $|\mu_\varepsilon^k| / N_\varepsilon \overset{*}{\rightharpoonup} \lambda_k \, dx$ for every k, and by property 3) we have

$$(77) \quad \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \int_\Omega W(\hat{\beta}_\varepsilon) \, dx = \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \int_{\Omega \setminus \Omega_{\rho_\varepsilon}(\mu_\varepsilon)} W(\hat{\beta}_\varepsilon) \, dx$$

$$= \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \sum_{k=1}^{M} |\mu_\varepsilon^k|(\Omega) \psi(\xi_k) = \sum_{k=1}^{M} \lambda_k \psi(\xi_k) = \varphi(\xi).$$

In particular, we deduce that $\hat{\beta}_\varepsilon / |\log \varepsilon|$ is bounded in $L^2(\Omega; \mathbb{M}^{2\times 2})$. Since the L^2 norm of $\hat{\beta}_\varepsilon / |\log \varepsilon|$ is concentrating on the hard core region, we conclude that $\hat{\beta}_\varepsilon / |\log \varepsilon|$ converges weakly to zero in

$L^2(\Omega; \mathbb{M}^{2\times 2})$. On the other hand, one can check directly that $\tilde{K}_\varepsilon^{\mu_\varepsilon}/|\log \varepsilon|$ converges strongly to zero in $L^2(\Omega)$. Recalling that also $R_\varepsilon/|\log \varepsilon| \to 0$, by (75) and (76) we conclude that $i)$ holds.

Next we prove $ii)$, namely, that the pair $(\beta_\varepsilon, \mu_\varepsilon)$ is optimal in energy. We have

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \int_\Omega W(\beta_\varepsilon) \, dx = \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \int_\Omega W(|\log \varepsilon|\beta + \hat{\beta}_\varepsilon) \, dx.$$

Since $\hat{\beta}_\varepsilon/|\log \varepsilon| \rightharpoonup 0$ in $L^2(\Omega; \mathbb{M}^{2\times 2})$, taking into account also (77), we conclude

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \int_\Omega W(\beta_\varepsilon) \, dx = \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|^2} \left(\int_\Omega W(|\log \varepsilon|\beta) \, dx + \int_\Omega W(\hat{\beta}_\varepsilon) \, dx \right) = \int_\Omega (W(\beta) + \varphi(\xi)) \, dx.$$

Finally, by the Lipschitz continuity of $\partial\Omega$, from (75) and (76) we also deduce that $(\beta_\varepsilon/|\log \varepsilon| - \beta) \cdot t$ tends to zero strongly in $H^{-\frac{1}{2}}(\partial\Omega)$.

Step 2. *The case* $\mu := \sum_{l=1}^L \chi_{A_l} \xi_l \, dx$.

In this step we proof the Γ-limsup inequality in the case of μ locally constant, i.e., of the type

$$(78) \qquad \mu := \sum_{l=1}^L \chi_{A_l} \xi_l \, dx,$$

where A_l are open subsets of Ω with Lipschitz continuous boundary and $\xi_i \in \mathbb{M}^{2\times 2}$. The construction of the recovery sequence is based on classical localization arguments in Γ-convergence and takes advantage of the previous step.

Let us set $\beta_l := \beta \llcorner A_l$, and let $\mu_{l,\varepsilon}$, $\beta_{l,\varepsilon}$ be the recovery sequence given by Step 1 applied with Ω replaced by A_l, with $\beta = \beta_l$, $\mu = \xi_l \, dx$. Finally let us set $\bar{\beta}_\varepsilon : \Omega \to \mathbb{M}^{2\times 2}$ and $\mu_\varepsilon \in \mathcal{M}(\Omega; \mathbb{R}^2)$ as follows

$$\bar{\beta}_\varepsilon(x) := \beta_{l,\varepsilon} \qquad \text{if } x \in A_l, \qquad \mu_\varepsilon := \sum_l \mu_{l,\varepsilon}.$$

By construction we have $\mu_\varepsilon \in X_\varepsilon$. Moreover since

$$\frac{1}{|\log \varepsilon|} \|\operatorname{Curl} \bar{\beta}_\varepsilon \llcorner \Omega_\varepsilon(\mu_\varepsilon)\|_{H^{-1}(\Omega; \mathbb{R}^2)} \le \sum_l \left\| \frac{\beta_{\varepsilon,l}}{|\log \varepsilon|} - \beta \right\|_{H^{-\frac{1}{2}}(\partial A_l)},$$

from the previous step it easily follows that

$$\frac{1}{|\log \varepsilon|} \operatorname{Curl} \bar{\beta}_\varepsilon \llcorner \Omega_\varepsilon(\mu_\varepsilon) \to 0$$

strongly in $H^{-1}(\Omega; \mathbb{R}^2)$. Therefore we can easily modify this sequence $\bar{\beta}_\varepsilon$ by adding a vanishing perturbation in order to obtain the desired recovery sequence β_ε.

Step 3. *The general case.*

In this step we show how to construct a recovery sequence in the general case (namely for a general dislocation measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^2)$).

Let (μ, β) be given in the domain of the Γ-limit $\mathcal{F}$. In view of the previous step and by standard density arguments in Γ-convergence, it is enough to construct sequences (μ_n, β_n) with $\operatorname{Curl} \beta_n = \mu_n$ and with μ_n locally constant as in (78) such that

$$(79) \qquad \beta_n \to \beta \text{ in } L^2(\Omega; \mathbb{M}^{2\times 2}), \qquad \mu_n \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{M}^{2\times 2}) \quad \text{and} \quad |\mu_n|(\Omega) \to |\mu|(\Omega).$$

Indeed, under these convergence assumptions we get the convergence of the corresponding energies, i.e.,

$$(80) \qquad \lim_{n \to +\infty} \mathcal{F}(\beta_n, \mu_n) = \mathcal{F}(\beta, \mu).$$

By standard reflection arguments we can extend the strain β to a function β_A defined in a neighborhood A of Ω, such that $\operatorname{Curl} \beta_A = \mu_A$ is a measure on A and moreover $|\mu_A|(\partial\Omega) = 0$.

Let ρ_h be a sequence of mollifiers, and define

$$f_h := \beta_A * \rho_h \llcorner \Omega, \qquad g_h := \mu_A * \rho_h \llcorner \Omega.$$

For h large enough these objects are well defined in Ω and $\operatorname{Curl} f_h = g_h$. Clearly

$$(81) \qquad f_h \to \beta \text{ in } L^2(\Omega; \mathbb{M}^{2\times 2}), \qquad g_h \, dx \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^2).$$

Moreover, since $|\mu_A|(\partial\Omega) = 0$, we have

$$(82) \qquad |g_h \, dx|(\Omega) \to |\mu|(\Omega).$$

Next, we approximate every g_h by locally constant functions, precisely, we consider a locally constant function $g_{h,k}$ such that

$$(83) \qquad |g_{h,k} - g_h|_{L^\infty(\Omega;\mathbb{R}^2)} \to 0 \text{ as } k \to \infty, \qquad \text{and} \qquad \int_\Omega g_{h,k} - g_h \, dx = 0 \,.$$

Let $r_{h,k}$ be the solution of the following problem

$$(84) \qquad \begin{cases} \operatorname{Curl} r_{h,k} = g_{h,k} - g_h & \text{in } \Omega, \\ \operatorname{Div} r_{h,k} = 0 & \text{in } \Omega, \\ r_{h,k} \cdot t = 0 & \text{in } \partial\Omega. \end{cases}$$

By standard elliptic estimates we have

$$(85) \qquad |r_{h,k}|_{L^2(\Omega;\mathbb{M}^{2\times 2})} \le C |g_{h,k} - g_h|_{L^2(\Omega;\mathbb{R}^2)}.$$

Finally, we set $f_{h,k} := f_h + r_{h,k}$. By (84) we have $\operatorname{Curl} f_{h,k} = g_{h,k}$. Moreover, by (83), (85) we have

$$(86) \qquad f_{h,k} \to f_h \text{ in } L^2(\Omega; \mathbb{M}^{2\times 2}) \qquad \text{as } k \to \infty.$$

By (81), (83), (86), using a diagonal argument we can find a sequence (μ_n, β_n) satisfying (79), and therefore (80).

7. The sub-critical case $(N_\varepsilon \ll |\log\varepsilon|)$

In this section we study the asymptotic behavior of the energy functionals E_ε defined in (19) in the case of dilute dislocations, i.e., for $N_\varepsilon \ll |\log\varepsilon|$. In terms of Γ-convergence, it means that we rescale E_ε with a prefactor $N_\varepsilon|\log\varepsilon|$, with $N_\varepsilon \ll |\log\varepsilon|$. As we discussed in Section 2, in this case the self-energy for minimizing sequences is predominant with respect to the interaction energy (see Remark 14).

In contrast to the critical case, we have that the prefactors of strains and dislocation measures in the sub-critical case are different. Indeed, the natural rescaling for the dislocation measures is given by N_ε. On the other hand, in order to catch the effect of the diffuse energy associated with a sequence $(\mu_\varepsilon, \beta_\varepsilon)$ with bounded energy we have to rescale the strains by $(N_\varepsilon|\log\varepsilon|)^{1/2}$. These two quantities clearly coincide only in the critical case $N_\varepsilon \equiv |\log\varepsilon|$. The effect of a different rescaling for strains and dislocation measures is that in the limit configuration μ and β are independent variables, i.e., the compatibility condition $\operatorname{Curl}\beta = \mu$ disappears in this limit. Actually the admissible strains in the limit are always gradients, i.e., $\operatorname{Curl}\beta = 0$. Heuristically this is a consequence of the fact that the total variation of $\operatorname{Curl}\beta_\varepsilon$ is of order N_ε, so that $\operatorname{Curl}(\beta_\varepsilon/(N_\varepsilon|\log\varepsilon|)^{1/2})$ vanishes.

The candidate Γ-limit of the functionals $\mathcal{F}_\varepsilon^{\text{dilute}} : \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2\times 2})$ defined in (21) is the functional $\mathcal{F}^{\text{dilute}}$ defined by

$$(87) \qquad \mathcal{F}^{\text{dilute}}(\mu, \beta) := \begin{cases} \displaystyle\int_\Omega W(\beta) \, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu| & \text{if } \operatorname{Curl}\beta = 0; \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{M}^{2\times 2}). \end{cases}$$

The precise Γ-convergence result is the following.

Theorem 13. *Let $N_\varepsilon \to \infty$ be such that $N_\varepsilon/|\log\varepsilon| \to 0$. Then the following Γ-convergence result holds.*

  i) **Compactness.** *Let $\varepsilon_n \to 0$ and let $\{(\mu_n, \beta_n)\}$ be a sequence in $\mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2\times 2})$ such that $\mathcal{F}_{\varepsilon_n}^{\text{dilute}}(\mu_n, \beta_n) \le E$ for some positive constant E independent of n. Then there exist $\mu \in \mathcal{M}(\Omega; \mathbb{R}^2)$ and $\beta \in L^2(\Omega; \mathbb{M}^{2\times 2})$, with $\operatorname{Curl}\beta = 0$, such that (up to a subsequence)*

$$(88) \qquad \frac{1}{N_{\varepsilon_n}}\mu_n \overset{*}{\rightharpoonup} \mu \qquad \text{in} \quad \mathcal{M}(\Omega; \mathbb{R}^2),$$

$$(89) \qquad \frac{1}{(N_{\varepsilon_n} |\log \varepsilon_n|)^{1/2}} \beta_n \rightharpoonup \beta \qquad \text{in} \quad L^2(\Omega; \mathbb{M}^{2\times 2}).$$

ii) **Γ-convergence.** *The functionals $\mathcal{F}_\varepsilon$ Γ-converge as $\varepsilon \to 0$, with respect to the convergence in (88) and (89), to the functional $\mathcal{F}^{\text{dilute}}$ defined in (87). More precisely, the following inequalities hold.*

Γ-liminf inequality: for every $(\mu, \beta) \in \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2\times 2})$, with $\mathrm{Curl}\,\beta = 0$, and for every sequence $(\mu_\varepsilon, \beta_\varepsilon) \in X_\varepsilon \times L^2(\Omega; \mathbb{M}^{2\times 2})$ satisfying (88) and (89), we have

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon^{\text{dilute}}(\mu_\varepsilon, \beta_\varepsilon) \geq \mathcal{F}^{\text{dilute}}(\mu, \beta);$$

Γ-limsup inequality: given $(\mu, \beta) \in \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2\times 2})$, with $\mathrm{Curl}\,\beta = 0$, there exists $(\mu_\varepsilon, \beta_\varepsilon) \in X_\varepsilon \times L^2(\Omega; \mathbb{M}^{2\times 2})$ satisfying (88) and (89) such that

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon^{\text{dilute}}(\mu_\varepsilon, \beta_\varepsilon) \leq \mathcal{F}^{\text{dilute}}(\mu, \beta).$$

Remark 14. The independence of strains and dislocation measures in the Γ-limit is a consequence of the fact that, in the dilute regime, the interaction energy is a lower order term with respect to the self-energy. Indeed, by the Γ-convergence result stated in Theorem 13 we immediately deduce that the functionals $\mathcal{E}_\varepsilon^{\text{dilute}}$, defined by

$$\mathcal{E}_\varepsilon^{\text{dilute}}(\mu) := \min_{\beta \in \mathcal{AS}_\varepsilon(\Omega)} \mathcal{F}_\varepsilon^{\text{dilute}}(\mu, \beta),$$

Γ-converge (as $\varepsilon \to 0$) to the functional $\mathcal{E}^{\text{dilute}} : \mathcal{M}(\Omega; \mathbb{R}^2) \to \mathbb{R}$ defined by

$$\mathcal{E}^{\text{dilute}}(\mu) := \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|} \right) d|\mu|.$$

The energy $\mathcal{E}^{\text{dilute}}(\mu)$ represents the energy stored in the crystal induced by the distribution of dislocations μ in the dilute regime, and it is given only by the self-energy.

The proof of Theorem 13 follows the lines of the proof of Theorem 10. For the reader convenience we sketch its main steps.

Proof of Theorem 13. The compactness property of rescaled strains and dislocation measures stated in (88) and (89) can be proved with minor changes as in the critical case. Let us prove that in this case $\mathrm{Curl}\,\beta = 0$. Let $\varphi \in C_0^1(\Omega)$ and let $\{\varphi_n\} \subset H_0^1(\Omega)$ be a sequence converging to φ uniformly and strongly in $H_0^1(\Omega)$ and satisfying the property

$$\varphi_n \equiv \varphi(x_{i,n}) \qquad \text{in } B_{\varepsilon_n}(x_{i,n}) \quad \text{for every } x_{i,n} \text{ in the support set of } \mu_n.$$

By Remark 1 we have

$$< \mathrm{Curl}\,\beta, \varphi > = \lim_{\varepsilon_n \to 0} \frac{1}{N_{\varepsilon_n}^{1/2} |\log \varepsilon_n|^{1/2}} < \mathrm{Curl}\,\beta_n, \varphi_n >$$

$$= \lim_{\varepsilon_n \to 0} \frac{N_{\varepsilon_n}^{1/2}}{|\log \varepsilon_n|^{1/2}} \frac{1}{N_{\varepsilon_n}} \int_\Omega \varphi_n \, d\mu_n = \lim_{\varepsilon_n \to 0} \frac{N_{\varepsilon_n}^{1/2}}{|\log \varepsilon_n|^{1/2}} \int_\Omega \varphi \, d\mu = 0$$

from which we deduce $\mathrm{Curl}\,\beta = 0$.

Concerning the Γ-convergence result, the proof of Γ-liminf inequality is identical to that of the critical case, so that we pass directly to the proof of the Γ-limsup inequality.

As in the critical case, classical localization arguments reduce the problem to the case of μ constant. (Note that the density argument used in the critical case is even easier in the sub-critical case, since no admissibility condition $\mathrm{Curl}\,\beta = \mu$ is required.)

The proof of the Γ-limsup inequality reduces to find a sequence $\{\mu_\varepsilon\} \subset X_\varepsilon$, with $1/N_\varepsilon \mu_\varepsilon \overset{*}{\rightharpoonup} \xi\, dx$ in $\mathcal{M}(\Omega; \mathbb{R}^2)$, and a sequence $\beta_\varepsilon \in \mathcal{AS}_\varepsilon(\mu_\varepsilon)$, with $1/(N_{\varepsilon_n}|\log \varepsilon_n|)^{1/2} \beta_\varepsilon \rightharpoonup \beta$ in $L^2(\Omega; \mathbb{M}^{2\times 2})$, such that

$$(90) \qquad \limsup_{\varepsilon \to 0} \frac{1}{N_{\varepsilon_n} |\log \varepsilon_n|} \int_\Omega W(\beta_\varepsilon) \, dx \leq \int_\Omega (W(\beta) + \varphi(\xi)) \, dx,$$

and satisfying the additional requirement that $(\beta_\varepsilon/|\log \varepsilon| - \beta) \cdot t$ tends to zero strongly in $H^{-\frac{1}{2}}(\partial \Omega)$.

Consider the sequence $\mu_\varepsilon := \sum_{i=1}^{M_\varepsilon} \xi_{\varepsilon,i} \delta_{x_{\varepsilon,i}}$ given by Lemma 12. Construct the functions $\hat{\beta}_\varepsilon : \Omega \to \mathbb{R}^2$ as in (74). Then set

$$(91) \qquad \bar{\beta}_\varepsilon := (N_\varepsilon |\log \varepsilon|)^{\frac{1}{2}} \beta - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \hat{\beta}_\varepsilon,$$

where $\tilde{K}_\varepsilon^{\mu_\varepsilon}$ is defined according to (67). By (68)

$$\operatorname{Curl} \bar{\beta}_\varepsilon \llcorner \Omega_\varepsilon(\mu_\varepsilon) = -\tilde{\mu}_\varepsilon^{r_\varepsilon}$$

By its definition the density of the measure $\tilde{\mu}_\varepsilon^{r_\varepsilon}/(N_\varepsilon |\log \varepsilon|)^{\frac{1}{2}}$ tends to zero uniformly as $\varepsilon \to 0$. Therefore, we can add to $\bar{\beta}_\varepsilon$ a sequence R_ε in $L^2(\Omega; \mathbb{M}^{2\times 2})$, with $R_\varepsilon/(N_\varepsilon |\log \varepsilon|)^{\frac{1}{2}} \to 0$, obtaining the admissible strain

$$(92) \qquad \beta_\varepsilon := (\bar{\beta}_\varepsilon + R_\varepsilon)\chi_{\Omega_\varepsilon(\mu_\varepsilon)}.$$

In order to prove that the pair $(\mu_\varepsilon, \beta_\varepsilon)$ is the desired recovery sequence, we have to check the following properties

   i) β_ε converge to β in the sense of definition (89);
   ii) The pair $(\beta_\varepsilon, \mu_\varepsilon)$ is a recovery sequence, i.e.,

$$\lim_{\varepsilon \to 0} \int_\Omega W(\beta_\varepsilon)\, dx = \int_\Omega (W(\beta) + \varphi(\xi))\, dx.$$

To prove i), recalling that by Lemma 12, $|\mu_\varepsilon^k|/N_\varepsilon \stackrel{*}{\rightharpoonup} \lambda_k\, dx$ for every k, we have

$$(93) \qquad \lim_{\varepsilon \to 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \int_\Omega W(\hat{\beta}_\varepsilon)\, dx = \lim_{\varepsilon \to 0} \frac{1}{N_\varepsilon} \sum_{k=1}^M |\mu_\varepsilon^k|(\Omega)\psi(\xi_k) = \sum_{k=1}^M \lambda_k \psi(\xi_k) = \varphi(\xi).$$

We deduce that $\hat{\beta}_\varepsilon/(N_\varepsilon |\log \varepsilon|)^{1/2}$ is bounded in $L^2(\Omega; \mathbb{M}^{2\times 2})$. As in the critical case, the L^2 norm of $\hat{\beta}_\varepsilon/|\log \varepsilon|$ is concentrating on the hard core region, so that $\hat{\beta}_\varepsilon/|\log \varepsilon|$ converges weakly to zero in $L^2(\Omega; \mathbb{M}^{2\times 2})$. On the other hand one can check directly that $\tilde{K}_\varepsilon^{\mu_\varepsilon}/(N_\varepsilon |\log \varepsilon|)^{1/2}$ converges strongly to zero in $L^2(\Omega)$, from which property i) follows.

Concerning ii), we prove that the pair $(\beta_\varepsilon, \mu_\varepsilon)$ is optimal in energy. We have

$$\lim_{\varepsilon \to 0} \frac{1}{(N_\varepsilon |\log \varepsilon|)} \int_\Omega W(\beta_\varepsilon)\, dx = \lim_{\varepsilon \to 0} \frac{1}{(N_\varepsilon |\log \varepsilon|)} \int_\Omega W(|\log \varepsilon|\beta + \hat{\beta}_\varepsilon)\, dx.$$

Since $\hat{\beta}_\varepsilon/|\log \varepsilon| \rightharpoonup 0$ in $L^2(\Omega; \mathbb{M}^{2\times 2})$, taking into account also (93), we conclude

$$\lim_{\varepsilon \to 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \int_\Omega W(\beta_\varepsilon)\, dx = \lim_{\varepsilon \to 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \left(\int_\Omega W((N_\varepsilon |\log \varepsilon|)^{\frac{1}{2}}\beta)\, dx + \int_\Omega W(\hat{\beta}_\varepsilon)\, dx \right)$$
$$= \int_\Omega (W(\beta) + \varphi(\xi))\, dx.$$

Finally, by the Lipschitz continuity of $\partial\Omega$, from (91) and (92) we also deduce that $(\beta_\varepsilon/(N_\varepsilon |\log \varepsilon|)^{1/2} - \beta) \cdot t$ tends to zero strongly in $H^{-\frac{1}{2}}(\partial\Omega)$. $\qquad\square$

Remark 15. The case $N_\varepsilon \leq C$ has been considered in [6], where the asymptotic behavior of the elastic energy for a fixed distribution of dislocations $\mu_\varepsilon \equiv \mu$ is provided up to the second order, and in [21], where the problem of Γ-convergence induced by screw dislocations is addressed (with $E_\varepsilon(\beta) := \|\beta\|_2^2$), without any assumption involving the notion of hard core region (essentially with $\rho_\varepsilon \approx \varepsilon$).

We could extend the result given by Theorem 13 to the case $N_\varepsilon \leq C$, obtaining a Γ-limit which has still the form as in (87), but with $\mu := \sum_{i=1}^M \xi_i\, \delta_{x_i}$, where $\xi_i \in \mathbb{S}$, and with $\varphi : \mathbb{S} \to \mathbb{R}$ defined now by

$$\varphi(\xi) := \inf \left\{ \sum_{k=1}^N \psi(\xi_k) : \sum_{k=1}^N \xi_k = \xi,\ N \in \mathbb{N},\ \xi_k \in \mathbb{S} \right\}.$$

8. The super-critical case $(N_\varepsilon \gg |\log \varepsilon|)$

In this section we study the asymptotic behavior of the energy functionals E_ε defined in (19) in the super-critical case, i.e., for $N_\varepsilon \gg |\log \varepsilon|$. In terms of Γ-convergence, it means that we rescale E_ε by N_ε^2, obtaining the rescaled energy functionals $\mathcal{F}_\varepsilon^{\mathrm{super}}$ defined in (23). As we discussed in Section 2, in this case we have that the interaction energy for minimizing sequences is predominant with respect to the self-energy.

The natural rescaling for the strains in this case is given by N_ε, but we don't have any control on the total variation of the dislocation measure. As a consequence we will get a limit energy $\mathcal{F}^{\mathrm{super}}$ defined on $L^2(\Omega; \mathbb{M}_{\mathrm{sym}}^{2\times 2})$ depending on strains given by

$$(94) \qquad \mathcal{F}^{\mathrm{super}}(\beta^{\mathrm{sym}}) := \int_\Omega W(\beta^{\mathrm{sym}})\, dx \quad \text{if } \beta^{\mathrm{sym}} \in L^2(\Omega; \mathbb{M}_{\mathrm{sym}}^{2\times 2}),$$

where $\mathbb{M}_{\mathrm{sym}}^{2\times 2}$ denotes the class of symmetric matrices M^{sym} in $\mathbb{M}^{2\times 2}$. The precise Γ-convergence result is the following.

Theorem 16. *Let N_ε be such that $N_\varepsilon/|\log \varepsilon| \to \infty$ as $\varepsilon \to 0$. Then the following Γ-convergence result holds.*

   i) **Compactness.** *Let $\varepsilon_n \to 0$ and let $\{(\mu_n, \beta_n)\}$ be a sequence in $\mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{M}^{2\times 2})$ such that $\mathcal{F}_{\varepsilon_n}^{\mathrm{super}}(\mu_n, \beta_n) \le E$ for some positive constant E independent of n. Then there exists a strain $\beta^{\mathrm{sym}} \in L^2(\Omega; \mathbb{M}^{2\times 2})$, such that (up to a subsequence)*

$$(95) \qquad \frac{1}{N_{\varepsilon_n}} \beta_n^{\mathrm{sym}} \rightharpoonup \beta^{\mathrm{sym}} \quad in\ L^2(\Omega; \mathbb{M}_{sym}^{2\times 2}).$$

   ii) **Γ-convergence.** *The functionals $\mathcal{F}_\varepsilon^{\mathrm{super}}$ Γ-converge as $\varepsilon \to 0$, with respect to the convergence in (95), to the functional $\mathcal{F}^{\mathrm{super}}$ defined in (94). More precisely, the following inequalities hold.*

   Γ-liminf inequality: for every $\beta^{\mathrm{sym}} \in L^2(\Omega; \mathbb{M}_{sym}^{2\times 2})$ and for every sequence $(\mu_\varepsilon, \beta_\varepsilon) \in X_\varepsilon \times L^2(\Omega; \mathbb{M}^{2\times 2})$ satisfying (95) we have

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon^{\mathrm{super}}(\mu_\varepsilon, \beta_\varepsilon) \ge \mathcal{F}^{\mathrm{super}}(\beta^{\mathrm{sym}});$$

   Γ-limsup inequality: given $\beta^{\mathrm{sym}} \in L^2(\Omega; \mathbb{M}_{sym}^{2\times 2})$ there exists $(\mu_\varepsilon, \beta_\varepsilon) \in X_\varepsilon \times L^2(\Omega; \mathbb{M}^{2\times 2})$ satisfying (95) such that

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon^{\mathrm{super}}(\mu_\varepsilon, \beta_\varepsilon) \le \mathcal{F}^{\mathrm{super}}(\beta^{\mathrm{sym}}).$$

Proof of Theorem 16. The compactness property is simply due to the usual apriori L^2 bound for the strains $\beta_{\varepsilon_n}^{\mathrm{sym}}$, while the Γ-liminf inequality comes simply by lower semicontinuity.

The main difference with respect to the previous energy regimes is in the proof of the Γ-limsup inequality. Again the strategy is to approximating a special class of limiting configurations and then to proceed by density, but in this case it will be more convenient to approximate the strains β with C^1 functions, so that their Curl's are continuous.

Thus, fix $\beta \in L^2(\Omega; \mathbb{M}^{2\times 2})$ such that $\mathrm{Curl}\,\beta$ is a measure μ of the type $\mu = g(x)\, dx$ with g continuous and let us construct a sequence $\{\mu_\varepsilon\} \subset X_\varepsilon(\Omega)$ and a sequence $\beta_\varepsilon \in \mathcal{AS}_\varepsilon(\mu_\varepsilon)$, with $\beta_\varepsilon/N_\varepsilon \rightharpoonup \beta$ in $L^2(\Omega; \mathbb{M}^{2\times 2})$, such that the following Γ-limsup inequality holds true

$$(96) \qquad \limsup_{\varepsilon \to 0} \frac{1}{N_\varepsilon^2} \int_\Omega W(\beta_\varepsilon)\, dx \le \int_\Omega W(\beta)\, dx.$$

Arguing as in the proof of Lemma 12, it is easy to prove that there exist $C \in \mathbb{R}$ depending only on $\|g\|_{L^\infty(\Omega; \mathbb{R}^2)}$ and a sequence of measures $\mu_\varepsilon := \sum_{i=1}^{M_\varepsilon} \xi_{i,\varepsilon} \delta_{x_{i,\varepsilon}} \in X_\varepsilon$, with $|\xi_{i,\varepsilon}| \le C$, such that, setting $r_\varepsilon := C/\sqrt{N_\varepsilon}$, we have $B_{r_\varepsilon}(x_{i,j}) \in \Omega$, $|x_{i,\varepsilon} - x_{j,\varepsilon}| \ge 2r_\varepsilon$ for every $x_{i,\varepsilon}$, $x_{j,\varepsilon}$ in the support set of μ_ε, and, finally,

$$\frac{\mu_\varepsilon}{N_\varepsilon} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^2), \quad \frac{\hat{\mu}_\varepsilon^{r_\varepsilon}}{N_\varepsilon} \to \mu \text{ strongly in } H^{-1}(\Omega; \mathbb{R}^2),$$

where $\hat{\mu}_\varepsilon$ is defined according to (65). Consider the functions $\hat{K}^{\mu_\varepsilon}_\varepsilon$ defined in (67) and set $\bar{\beta}_\varepsilon := N_\varepsilon \beta + \hat{K}^{\mu_\varepsilon}_\varepsilon$. By (68) we deduce

$$\text{Curl} \frac{\bar{\beta}_\varepsilon}{N_\varepsilon} \llcorner \Omega_\varepsilon(\mu_\varepsilon) = (\mu - \frac{\hat{\mu}^{r_\varepsilon}_\varepsilon}{N_\varepsilon}) \to 0 \quad \text{in } H^{-1}(\Omega; \mathbb{R}^2).$$

Therefore, we can add to $\bar{\beta}_\varepsilon$ a vanishing sequence $\frac{R_\varepsilon}{N_\varepsilon} \to 0$ in $L^2(\Omega; \mathbb{M}^{2\times 2})$, obtaining the admissible strain

$$\beta_\varepsilon := (\bar{\beta}_\varepsilon + R_\varepsilon) \chi_{\Omega_\varepsilon(\mu_\varepsilon)}.$$

In order to prove that the pair $(\mu_\varepsilon, \beta_\varepsilon)$ is the desired recovery sequence it is enough to observe that $\hat{K}^{\mu_\varepsilon}_\varepsilon \to 0$ in $L^2(\Omega; \mathbb{M}^{2\times 2})$. Indeed, by construction we have $M_\varepsilon \leq C N_\varepsilon$, and therefore

$$\lim_{\varepsilon \to 0} \frac{1}{N^2_\varepsilon} \int_{\Omega_\varepsilon} |\hat{K}^{\mu_\varepsilon}_\varepsilon|^2 \, dx \leq \lim_{\varepsilon \to 0} \frac{C}{N^2_\varepsilon} M_\varepsilon |\log \varepsilon| \leq \lim_{\varepsilon \to 0} C \frac{|\log \varepsilon|}{N_\varepsilon} = 0.$$

$\square$

Remark 17. Note that in the super-critical regime we can not have a compactness property for the antisymmetric part of the admissible strains β_ε, and indeed it is easy to exhibit examples where $\|\beta^{\text{sym}}_\varepsilon / N_\varepsilon\|_{L^2(\Omega; \mathbb{M}^{2\times 2}_{\text{sym}})} \leq C$ and $\|\beta^{\text{skew}}_\varepsilon / N_\varepsilon\|_{L^2(\Omega; \mathbb{M}^{2\times 2})} \to \infty$. Note that, since we do not have any control on the mass of $\text{Curl} \, \beta_\varepsilon$, we can not apply Theorem 9.

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